Poornima College of Engineering, Jaipur

Course CO-PO, Preparation, Assessment, Attainments
Academic Session: 2018-2019

Section:4CSA Semester- IV
Subject: SPT Subject Code: 4CS3

Name of the Faculty: Dr. Shilpi Jain

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Syllabus

Rajasthan Technical University, Kota

STATISTICS & PROBABILITY THEORY

For B.Tech. 4th Sem. (CS/IT)

UNIT	CONTENTS
UNIT- I	Introduction & Discrete random variables Sample space, events, algebra of events, Bernoulli's trials, Probability & Baye's theorem. Random variable & their event space, probability generating function, expectations, moments, computations of mean time to failure, Bernoulli & Poisson processes.
UNIT- II	Discrete & continuous distributions Probability distribution & probability densities: Binomial, Poisson, normal rectangular and exponential distribution & their PDF's, moments and MGF's for above distributions.
UNIT- III	Correlation & Regression Correlation & regression: Linear regression, Rank correlation, Method of least squares Fitting of straight lines & second degree parabola. Normal regression and correlation analysis.
UNIT- IV	Queuing Theory Pure birth, pure death and birth-death processes. Mathematical models for M/M/1, M/M/N, M/M/S and M/M/S/N queues.
UNIT- V	Discrete Parameter mark on chains: M/G/1 Queuing model, Discrete parameter birth-death process.

Poornima College of Engineering, Jaipur

Department of Computer Engineering

Vision

Evolve as a centre of excellence with wider recognition and to adapt the rapid innovation in Computer Engineering

Mission

- To provide a learning-centered environment that will enable students and faculty members to achieve their goals empowering them to compete globally for the most desirable careers in academia and industry
- To contribute significantly to the research and the discovery of new arenas of knowledge and methods in the rapid developing field of Computer Engineering
- To support society through participation and transfer of advanced technology from one sector to another

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Poornima College of Engineering, Jaipur
Department of Computer Engineering

Program Educational Objectives (PEOs)

- Graduates will work productively as skillful engineers playing the leading roles in multifaceted teams.
- Graduates will identify the solutions for challenging issues inspiring the upcoming generations leading them towards innovative, creative, and sophisticated technologies.
- Graduates will implement their pioneering ideas practically to create novel products and the feasible solutions of research oriented problems

Program Outcomes (POs)

a. An ability to apply the knowledge of mathematics, science, engineering fundamentals, algorithmic principles and computing for the solution of complex engineering problems.

b. An ability to design and conduct experiments, analyze and interpret data finding the computing requirements with appropriate solutions.

c. An ability to design a computer based system, component, or process to meet the desired needs within realistic constraints such as economic, environmental, social, political, ethical, health, safety, manufacturability and sustainability etc.

d. An ability to function effectively in multidisciplinary teams,

e. An ability to identify, formulate, and solve engineering problems by applying the appropriate computing techniques, progressive resources and modern IT tools with an understanding of the limitations.

f. An understanding of professional, ethical, security, legal, social issues and responsibilities towards these.

An ability to communicate effectively on complex engineering activities with the engineering community and the society.

h. The widen education necessary to understand and analyze the impact of computing and engineering solutions in a global, economic, environmental and societal context.

i. Recognition of the need with an engagement in lifelong learning in the context of technological change.

j. Knowledge of contemporary issues.

k. An ability to use the techniques, skills, and modern engineering tools necessary for computing and engineering practice.

l. An ability to contribute in the fruitful working as an efficient member of a team in multidisciplinary environments with a concern of engineering and management principles.

m. An ability to apply the basic principles and practices of Computer Engineering and its allied areas to fulfill the requirements of customers in context of business.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

TEACHING & LEARNING

Teaching learning process consists of four basic elements:

(a)**Assessment:** Conducting OBT, Assignments, tutorials, class test.

(b) **Planning:** Prepare the deployment, lecturer notes with the help of reference books and last year question papers. Also plan the SPL, PPT and Important question banks.

(c)**Implementation:** Through lecturer class, tutorials and extra lecturers.

(d) **Evaluation:** Two mid Term test and RTU Examination.

It is method for monitoring and judging the overall quality of learning or teaching based on objective, data and scientific criteria.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Poornima College of Engineering, Jaipur
Department of Computer Engineering
Session 2016-17(Even Semester)
RTU Syllabus - ABC Analysis

Sub Code: 4CS03
Sub Name: Statistics and probability Theory

Unit No.	A	B	C
1	Introduction & Discrete random variables Sample space, events, algebra of events, Bernoulli's trials, Probability & Baye's theorem	Random variable & their event space, probability generating function, expectations, moments, computations of mean time to failure,	Bernoulli & Poisson Processes.
2	Probability distribution & probability densities: Binomial, Poisson, normal rectangular and exponential distribution & their PDF's, moments and MGF's for above distributions.		
3			Correlation & Regression Correlation & regression: Linear regression, Rank correlation, Method of least squares Fitting of straight lines & second degree parabola. Normal regression and correlation analysis.
4		Queuing Theory Pure birth, pure death and birth-death processes. Mathematical models for M/M/1, M/M/N, M/M/S and M/M/S/N queues.	
5	Discrete Parameter mark on chains: M/G/1 Queuing model, Discrete parameter birth-death process.		

A – Difficult B - Average C – Easy

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Calendar for Even Semester (2017-2018)

January

Sun	Mon	Tue	Wed	Thu	Fri	Sat
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Aarohan

February

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28			

March

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Ist Midterm

April

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

IInd Midterm

May

Sun	Mon	Tue	Wed	Thu	Fri	Sat
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

June

Sun	Mon	Tue	Wed	Thu	Fri	Sat
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

Note: - Please do the needful.

1) Indicate/Underline the Mid-Term Exams dates as per Academic Calendar (AC).
2) Encircle the holidays as per AC.
3) Calculate the number of days available for actual conduction of classes as per time table provided by HOD.
4) If you are pursuing higher studies please plan for replacement of classes.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Start from 5 Jan. 2018

28 April 2018.

Section CSB

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Calendar for Even Semester (2017-2018)

Holiday

January

Sun	Mon	Tue	Wed	Thu	Fri	Sat
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

Holiday

February

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28			

Aarohan

March

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Holi Holiday

Ist Midterm

April

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

Holiday

May

Sun	Mon	Tue	Wed	Thu	Fri	Sat
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

IInd Midterm

June

Sun	Mon	Tue	Wed	Thu	Fri	Sat
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

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B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

II

POORNIMA
COLLEGE OF ENGINEERING

Calendar for Even Semester (2017-2018)

January

Sun	Mon	Tue	Wed	Thu	Fri	Sat
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	~~26~~	~~27~~
28	29	30	~~31~~			

February

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	2	3
~~4~~	~~5~~	~~6~~	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28			

Aarohan

March

Sun	Mon	Tue	Wed	Thu	Fri	Sat
				1	~~2~~	~~3~~
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Ist Midterm

April

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	5	6	7
8	9	10	11	12	13	~~14~~
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	~~30~~					

IInd Midterm

May

Sun	Mon	Tue	Wed	Thu	Fri	Sat
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

June

Sun	Mon	Tue	Wed	Thu	Fri	Sat
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

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Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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BLOWN UP SYLLABUS

Campus: PCE	**Course: B.tech.**	**Class/Section: II Yr**	**Date:03/01/2018**
Name of Faculty: Dr. Shilpi Jain		**Name of Subject: SPT**	**Code: 4CS03**

.No.	Topic as per Syllabus	BLOWN UP TOPICS (Up to 10 TIMES SYLLABUS)
	UNIT 1: DISCRETE RANDOM VARIABLES	
1.1	Sample space	1.1.1 Introduction to Probability 1.1.2 Deterministic Experiments 1.1.3 Random Experiments 1.1.4 Trial 1.1.5 Definition o f Sample space
1.2	Events &Algebra of Events	1.2.1.exhaustive events 1.2.2 Mutually exhaustive events 1.2.3 Union intersection & complement of events 1.2.4 laws of events 1.2.5 Probability 1.2.6 Conditional Probability
1.3	Baye's Theorem	1.3.1 Theorem o f Total Probability 1.3.2 Statement & proof of Baye's Theorem
1.4	Random Variables	1.4.1 Introduction 1.4.2 Definition 1.4.2 probability Mass Function 1.4.3 Discrete & Continuous random variables 1.4.4 Probability generating Function 1.4.5 Bivariate Random Variables
1.5	Mathematical Expectation	1.5.1 Moments 1.5.2 Mathematical Expectation 1.5.3 Computation of Mean tin[illegible]

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
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1.6	Bernoulli Process	1.6.1 introduction 1.6.2 Definition o f random process 1.6.3 classification of random process 1.6.4 Markov process 1.6.6 Mean 1.6.7 Variance
1.7	Poisson process	1.7.1 concept 1.7.2 Poisson counting process 1.7.3 Homogenous Poisson process 1.7.4 Inter arrival time of Poisson process
	UNIT 2: **DISCRETE & CONTINUOUS DISTRIBUTIONS**	
2.1	Binomial Distribution	2.1.1 Introduction 2.1.2 Definition 2.1.3 p.m.f 2.1.4 mean 2.1.5 Variance 2.1.6 Moment Generating function
2.2	Poisson Distribution	2.2.1 Introduction 2.2.2 Definition 2.2.3 p.m.f 2.2.4 mean 2.2.5 Variance 2.2.6 Moment Generating function
2.3	Normal Distribution	2.3.1 Introduction 2.3.2 Definition 2.3.3 p.d.f 2.3.4 mean 2.3.5 Variance 2.3.6. Moment Generating function
2.4	Rectangular Distribution	2.4.1 Introduction 2.4.2 Definition 2.4.3 p.d.f 2.4.4 mean 2.4..5 Variance 2.4.6. Moment Generating function
2.5	Exponential distribution	2 .5.1 Introduction 2.5.2 Definition

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

		2.5.3 p.d.f 2.5.4 mean 2.5..5 Variance 2.5.6. Moment Generating function
	UNIT 3: **CORRELATION & REGRESSION**	
3.1	Linear regression	3.1.1 Introduction 3.1.2 Definition 3.1.3 Regression Equation & Some Theorems
3.2	Method of least square	3.2.1 Concept of Method 3.2.2 Fitting a Straight line & Parabola
3.3	Normal regression analysis	3.3.1 Concept & Derivation
3.4	Normal Correlation analysis	3.4.1 Concept & Derivation
	UNIT 4: **QUEUEING THEORY**	
4.1	Pure Birth, Pure Death & Birth- Death Process	4.1.1 Introduction 4.1.2 Queueing system 4.1.3 FIFO & LIFO 4.1.4 Probability distribution of arrival 4.1.5 Probability distribution of Waiting time 4.1.6 Birth Death Process (concept) 4.1.7 Differential-difference equation for Birth Death process 4.1.8 Steady state solution of Birth Death process 4.1.9 Pure Birth Process 4.1.10 Pure Death Process 4.1.11 Solution of Differential-difference equations
4.2	Mathematical Queueing Models(Model-I)	4.2.1 Concept & Basic Definition 4.2.2 Model I (M/M/I:8/FIFO) 4.2.3 Service time Distribution 4.2.4 pdf for waiting time in the system 4.2.5 pdf for waiting time in the queue 4.2.6 Average number of Customers in the System 4.2.7 Average waiting time of a customer in the system 4.2.9 Little Formula
4.3	Queueing Model II (M/M/I:N/FIFO)	4.3.1 Concept & Basic Definition 4.3.2 pdf for waiting time in the system

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

		4.3.3 pdf for waiting time in the system 4.3.4 pdf for waiting time in the queue 4.3.5 Average number of Customers in the System 4.3.6 Average waiting time of a customer in the system
4.4	Queueing Model III (M/M/S: ∞ /FIFO)	4.4.1 Concept 4.4.2 Average number of Customers in the queue 4.4.3 Average number of Customers in the System 4.4.4 Average waiting time of a customer in the system
4.5	Queueing Model IV (M/M/S: K /FIFO)	4.5.1 State -Transition diagram 4.5.2 Steady state Probability 4.5.3 Average number of Customers in the queue 4.5.3 Average number of Customers in the System 4.5.4 Average waiting time of a customer in the system
	UNIT 5: **DISCRETE PARAMETER MARKOV CHAIN**	
5.1	M/G/I Queueing Model	5.1.1 Introduction 5.1.2 Definition 5.1.3 Markov Chain 5.1.4 Transition probability 5.1.5 Order of Markov Chain 5.1.6 Expected number o f customer in the system 5.1.7 Average time taken in the system 5.1.8 Average number of Customers in the queue 5.1.9 Average time taken in the queue
5.2	Discrete Parameter Birth- Death Process	5.2.1 Steady State Probabilities 5.2.2 Questions

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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COLLEGE OF ENGINEERING

SYLLABUS DEPLOYMENT

Campus: PCE Course: B.Tech.	Class/Section: II Yr			Date: 03/01/2018		
Name of Faculty: Dr. Shilpi Jain	Name of Subject: SPT			Code: 4CS03A (B)		

No.	Topic As Per Blownup Syllabus	Lect. No.	Planned Date	Actual Del. Date	Reason For Deviation	Ref. / Text Book With Page No.
0	**Zero Lecture** **UNIT 1:** **DISCRETE RANDOM VARIABLES**	L0	8/1	8/1		
1.1	**Introduction of Lecture** 1.1.1Introduction to probability 1.1.2 Deterministic experiments 1.1.3 Random Experiments 1.1.4 Trial 1.1.5Definition of sample space 1.1.6 exhaustive events 1.1.7 Mutually exhaustive events **Conclusion of Lecture**	L1	9/1	9/1		
1.2	**Introduction of Lecture** 1.2.1 Union intersection & complement of events 1.2.2 laws o f events 1.2.3 Probability 1.2.4 Conditional Probability	L2	10/1	9/1	Entra.	

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Conclusion of Lecture					
1.3	**Introduction of Lecture** 1.3.1 Theorem of Total probability 1.3.2 Statement & proof of Baye's Theorem **Conclusion of Lecture**	L3	13/1	10/1		
1.4	**Introduction of Lecture** 1.4.1 Introduction 1.4.2 Definition 1.4.3 probability Mass Function 1.4.4 Discrete & continuous random variables **Conclusion of Lecture**	L4	15/1	11/1	Extra.	
1.5	**Introduction of Lecture** 1.5.1 Probability generating Function 1.5.2 Bivariate Random variables **Conclusion of Lecture**	L5	16/1	13/1		
1.6	**Introduction of Lecture** 1.6.1 Moments 1.6.2 Mathematical Expectation 1.6.3 Computation of Mean time to Failure **Conclusion of Lecture**	L6	17/1	15/1		
1.7	**Introduction of Lecture** 1.7.1 Introduction 1.7.2 Definition of random process 1.7.3 classification of random process **Conclusion of Lecture**	L7	20/1	16/1		
1.8	**Introduction of Lecture** 1.8.1 Markov process 1.8.2 Mean 1.8.3 Variance **Conclusion of Lecture**	L8	22/1	17/1		
1.9	**Introduction of Lecture** 1.9.1 Concept 1.9.2 Poisson counting process	L9	23/1	20/1		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	1.9.3 Homogenous Poisson process 1.9.4 Inter arrival time of Poisson process **Conclusion of Lecture**					
.10	**Introduction of Lecture** 1.10.1 Probability Related Question **Conclusion of Lecture** **** Class Test/Special lecture/OBT/Quiz**	L10	24/1	22/1		
2.1	**UNIT 2:** **DISCRETE & CONTINUOUS DISTRIBUTION** **Introduction of Lecture** Binomial Distribution 2.1.1 Introduction 2.1.2 Definition 2.1.3 p.m.f 2.1.4 mean 2.1.5 Variance 2.1.6 Moment Generating function **Conclusion of Lecture**	L11	27/1	23/1		
2.2	**Introduction of Lecture** 2.2.1 Poisson Distribution 2.2.2 Definition 2.2.3 p.m.f. 2.2.4 mean **Conclusion of Lecture**	L12	29/1	24/1		
2.3	**Introduction of Lecture** Poisson Distribution 2.3.1 Variance 2.3.2 Moment Generating function **Conclusion of Lecture**	L13	30/1	25/1		
2.4	**Introduction of Lecture** Normal Distribution 2.4.1 introduction 2.4.2 Definition 2.4.3 p.d.f 2.4.4 mean	L14	6/2	27/1		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Conclusion of Lecture					
2.5	**Introduction of Lecture** Normal Distribution 2.5.1 Variance 2.5.2 Moment Generating function **Conclusion of Lecture**	L15	7/2	5/2		
2.6	**Introduction of Lecture** Rectangular Distribution 2.6.1 Introduction 2.6.2 Definition 2.6.3 p.d.f 2.6.4 mean 2.6..5 Variance **Conclusion of Lecture**	L16	10/2	6/2		
2.7	**Introduction of Lecture** Rectangular Distribution 2.7.1 Moment Generating function 2.7.2 Related Question **Conclusion of Lecture**	L17	12/2	7/2		
2.8	**Introduction of Lecture** Exponential distribution 2 .8.1 Introduction 2.8.2 Definition 2.8.3 p.d.f 2.8.4 mean 2.8..5 Variance 2.8.6. Moment Generating function **Conclusion of Lecture**	L18	13/2	8/2	Extra.	
	**** Class Test/Special lecture/OBT/Quiz**	L19	14/2	10/2		
3.1	**UNIT-3:** **CORRELATION & REGRESSION** **Introduction of Lecture** Correlation 3.1.1 Introduction	L20	17/2	12/2		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	3.1.2 Karl Pearson Coefficient 3.1.3 Rank Correlation **Conclusion of Lecture**					
.2	**Introduction of Lecture** 3.2.1 Curve Fitting 3.2.2 Concept of Method (Least Square Method) 3.2.3 Fitting a Straight line **Conclusion of Lecture**	L21	19/2	13/2		
.3	**Introduction of Lecture** 3.3.1 Fitting of a Parabola 3.3.2 Related Question 3.3.3 Fitting of Other Curves **Conclusion of Lecture**	L22	20/2	17/2		
3.4	**Introduction of Lecture** Regression 3.4.1 Linear regression 3.4.2 Introduction 3.4.3 Definition 3.4.4Regression Equation & Some Theorems **Conclusion of Lecture**	L23	21/2	19/2		
3.5	**Introduction of Lecture** 3.5.1 Standard Error of Estimate or Residual Variance 3.5.2 Coefficient of Determination 3.5.3 Normal Correlation Analysis **Conclusion of Lecture**	L24	24/2	20/2		
3.6	**Introduction of Lecture** 3.6.1 Normal Regression Analysis 3.6.2 Related Question **Conclusion of Lecture**	L25	26/2	23/2		
	**** Class Test/Special lecture/OBT/Quiz** Revision. **UNIT 4:** **QUEUEING THEORY**	R1 R2 R3 R4 R5	27/2	24/2 26/2 27/2 28/2 5/3 6/3		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
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Poornima College of Engineering
ISI-6, RIICO Institutional Area
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4.1	**Introduction of Lecture** 4.1.1 Introduction 4.1.2 Queueing system 4.1.3 FIFO & LIFO 4.1.4 Probability distribution of arrival **Conclusion of Lecture**	L26	28/2			
4.2	**Introduction of Lecture** 4.2.1 Probability distribution of Waiting time 4.2.2 Birth Death Process (Concept) 4.2.3 Notations **Conclusion of Lecture**	L27	5/3			
4.3	**Introduction of Lecture** 4.3.1 Pure Birth Process 4.3.2 Markovian property of Inter arrival time **Conclusion of Lecture**	L28	6/3			
4.4	**Introduction of Lecture** 4.4.1 Pure Death Process **Conclusion of Lecture**	L29	7/3			
4.5	**Introduction of Lecture** 4.5.1 Birth & Death Process 4.5.2 Kendall's Notation **Conclusion of Lecture**	L30	10/3			
4.6	**Introduction of Lecture** 4.6.1 Queueing Model I (M/M/1:∞/FIFO) 4.6.2 Service time Distribution **Conclusion of Lecture**	L31	19/4			
4.7	**Introduction of Lecture** Related Problem to Model -I **Conclusion of Lecture**	L32	20/4			
	** Class Test/Special lecture/OBT/Quiz	L33	21/4			
4.8	**Introduction of Lecture**					

Dr. Mahesh Bundele
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	Queueing model II (M/M/1:N/FIFO) **Conclusion of Lecture**	L34	24/3			
4.9	**Introduction of Lecture** Related Problem to Model II **Conclusion of Lecture**	L35	26/3			
.10	**Introduction of Lecture** Queueing Model III Generalisation of Model I (M/M/1:∞/FCFS) **Conclusion of Lecture**	L36	27/3			
.11	**Introduction of Lecture** Queueing Model IV (M/M/s:∞/FCFS) **Conclusion of Lecture**	L37	28/3			
4.12	**Introduction of Lecture** Queueing Model V (M/M/s:N/FCFS) **Conclusion of Lecture**	L38	31/3			
	** Class Test/Special lecture/OBT/Quiz	L39	2/4			
5.1	**UNIT-5:** **DISCRETE PARAMETER MARKOV CHAIN** **Introduction of Lecture** 5.1.1 Introduction 5.1.2 Definition 5.1.3 Markov Chain 5.1.4 Transition Probability **Conclusion of Lecture**	L40	3/4			
	PPT	L41	4/4			
5.2	**Introduction of Lecture** 5.2.1 Steady State Distribution **Conclusion of Lecture**	L42	7/4			

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Introduction of Lecture 5.2.2 Chap Man Kolmogorov Theorem **Conclusion of Lecture**	L43	9/4			
5.3	**Introduction of Lecture** 5.3.1 Classification of States of Markov Chain 5.3.2 Related Question **Conclusion of Lecture**	L44	10/4			
5.4	**Introduction of Lecture** 5.4.1 Queueing Model (M/G/1:∞/GD) **Conclusion of Lecture**	L45	11/4			
5	**Introduction of Lecture** 5.5.1 Discrete Parameter Birth Death Process **Conclusion of Lecture**	L46,L 47	16/4, 17/4			
	**** Class Test/Special lecture/OBT/Quiz**	L48 L49	18/4 21/4			

L50 23/4

L51 24/4

L52 25/4

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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SYLLABUS DEPLOYMENT

Campus: PCE Course: B.Tech.	Class/Section: II Yr	Date: 03/01/2018
Name of Faculty: Dr. Shilpi Jain	Name of Subject: SPT	

No.	Topic As Per Blownup Syllabus	Lect. No.	Planned Date	Actual Del. Date	Reason For Deviation	Ref. / Text Book With Page No.
0	**Zero Lecture** **UNIT 1:** **DISCRETE RANDOM VARIABLES**	L0	8/1	10/1		
1.1	**Introduction of Lecture** 1.1.1Introduction to probability 1.1.2 Deterministic experiments 1.1.3 Random Experiments 1.1.4 Trial 1.1.5Definition of sample space 1.1.6 exhaustive events 1.1.7 Mutually exhaustive events **Conclusion of Lecture**	L1	10/1	11/1		
1.2	**Introduction of Lecture** 1.2.1 Union intersection & complement of events 1.2.2 laws o f events 1.2.3 Probability 1.2.4 Conditional Probability	L2	11/1	12/1		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Conclusion of Lecture					
1.3	**Introduction of Lecture** 1.3.1 Theorem of Total probability 1.3.2 Statement & proof of Baye's Theorem **Conclusion of Lecture**	L3	12/1	15/1		
1.4	**Introduction of Lecture** 1.4.1 Introduction 1.4.2 Definition 1.4.3 probability Mass Function 1.4.4 Discrete & continuous random variables **Conclusion of Lecture**	L4	15/1	17/1		
1.5	**Introduction of Lecture** 1.5.1 Probability generating Function 1.5.2 Bivariate Random variables **Conclusion of Lecture**	L5	17/1	18/1		
1.6	**Introduction of Lecture** 1.6.1 Moments 1.6.2 Mathematical Expectation 1.6.3 Computation of Mean time to Failure **Conclusion of Lecture**	L6	18/1	19/1		
1.7	**Introduction of Lecture** 1.7.1 Introduction 1.7.2 Definition of random process 1.7.3 classification of random process **Conclusion of Lecture**	L7	19/1	22/1		
1.8	**Introduction of Lecture** 1.8.1 Markov process 1.8.2 Mean 1.8.3 Variance **Conclusion of Lecture**	L8	22/1	24/1		
1.9	**Introduction of Lecture** 1.9.1 Concept 1.9.2 Poisson counting process	L9	24/1	25/1		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	1.9.3 Homogenous Poisson process 1.9.4 Inter arrival time of Poisson process **Conclusion of Lecture**					
.10	**Introduction of Lecture** 1.10.1 Probability Related Question **Conclusion of Lecture** **** Class Test/Special lecture/OBT/Quiz**	L10	25/1	5/2		
2.1	**UNIT 2:** **DISCRETE & CONTINUOUS DISTRIBUTION** **Introduction of Lecture** Binomial Distribution 2.1.1 Introduction 2.1.2 Definition 2.1.3 p.m.f 2.1.4 mean 2.1.5 Variance 2.1.6 Moment Generating function **Conclusion of Lecture**	L11	29/1	7/2		
2.2	**Introduction of Lecture** 2.2.1 Poisson Distribution 2.2.2 Definition 2.2.3 p.m.f. 2.2.4 mean **Conclusion of Lecture**	L12	7/2	8/2		
2.3	**Introduction of Lecture** Poisson Distribution 2.3.1 Variance 2.3.2 Moment Generating function **Conclusion of Lecture**	L13	8/2	9/2		
2.4	**Introduction of Lecture** Normal Distribution 2.4.1 introduction 2.4.2 Definition 2.4.3 p.d.f 2.4.4 mean	L14	9/2	12/2		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Conclusion of Lecture					
2.5	**Introduction of Lecture** Normal Distribution 2.5.1 Variance 2.5.2 Moment Generating function **Conclusion of Lecture**	L15	12/2	14/2		
2.6	**Introduction of Lecture** Rectangular Distribution 2.6.1 Introduction 2.6.2 Definition 2.6.3 p.d.f 2.6.4 mean 2.6..5 Variance **Conclusion of Lecture**	L16	14/2	15/2		
2.7	**Introduction of Lecture** Rectangular Distribution 2.7.1 Moment Generating function 2.7.2 Related Question **Conclusion of Lecture**	L17	15/2	16/2		
2.8	**Introduction of Lecture** Exponential distribution 2 .8.1 Introduction 2.8.2 Definition 2.8.3 p.d.f 2.8.4 mean 2.8..5 Variance 2.8.6. Moment Generating function **Conclusion of Lecture**	L18	16/2	19/2		
	**** Class Test/Special lecture/OBT/Quiz**	L19	19/2	21/2		
3.1	**UNIT-3:** **CORRELATION & REGRESSION** **Introduction of Lecture** Correlation 3.1.1 Introduction	L20	21/2	22/2		

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	3.1.2 Karl Pearson Coefficient 3.1.3 Rank Correlation **Conclusion of Lecture**					
3.2	**Introduction of Lecture** 3.2.1 Curve Fitting 3.2.2 Concept of Method (Least Square Method) 3.2.3 Fitting a Straight line **Conclusion of Lecture**	L21	22/2	23/2		
3.3	**Introduction of Lecture** 3.3.1 Fitting of a Parabola 3.3.2 Related Question 3.3.3 Fitting of Other Curves **Conclusion of Lecture**	L22	23/2	28/2		
3.4	**Introduction of Lecture** Regression 3.4.1 Linear regression 3.4.2 Introduction 3.4.3 Definition 3.4.4Regression Equation & Some Theorems **Conclusion of Lecture**	L23	26/2	5/3		
3.5	**Introduction of Lecture** 3.5.1 Standard Error of Estimate or Residual Variance 3.5.2 Coefficient of Determination 3.5.3 Normal Correlation Analysis **Conclusion of Lecture**	L24	28/2	7/3		
3.6	**Introduction of Lecture** 3.6.1 Normal Regression Analysis 3.6.2 Related Question **Conclusion of Lecture**	L25	5/3	7/3	Extra.	
	**** Class Test/Special lecture/OBT/Quiz**		7/3	8/3		
	Revesion		[illegible]	9/3		
	UNIT 4: **QUEUEING THEORY**					

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

4.1	**Introduction of Lecture** 4.1.1 Introduction 4.1.2 Queueing system 4.1.3 FIFO & LIFO 4.1.4 Probability distribution of arrival **Conclusion of Lecture**	L26	8/3			
4.2	**Introduction of Lecture** 4.2.1 Probability distribution of Waiting time 4.2.2 Birth Death Process (Concept) 4.2.3 Notations **Conclusion of Lecture**	L27	9/3			
4.3	**Introduction of Lecture** 4.3.1 Pure Birth Process 4.3.2 Markovian property of Inter arrival time **Conclusion of Lecture**	L28	19/3			
4.4	**Introduction of Lecture** 4.4.1 Pure Death Process **Conclusion of Lecture**	L29	21/3			
4.5	**Introduction of Lecture** 4.5.1 Birth & Death Process 4.5.2 Kendall's Notation **Conclusion of Lecture**	L30	22/3			
4.6	**Introduction of Lecture** 4.6.1 Queueing Model I (M/M/1:∞/FIFO) 4.6.2 Service time Distribution **Conclusion of Lecture**	L31	23/3			
4.7	**Introduction of Lecture** Related Problem to Model -I **Conclusion of Lecture**	L32	26/3			
	** Class Test/Special lecture/OBT/Quiz	L33	28/3			
4.8	**Introduction of Lecture**					

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Queueing model II (M/M/1:N/FIFO) **Conclusion of Lecture**	L34	19/3			
4.9	**Introduction of Lecture** Related Problem to Model II **Conclusion of Lecture**	L35	30/3			
4.10	**Introduction of Lecture** Queueing Model III Generalisation of Model I (M/M/1:∞/FCFS) **Conclusion of Lecture**	L36	2/4			
4.11	**Introduction of Lecture** Queueing Model IV (M/M/s:∞/FCFS) **Conclusion of Lecture**	L37	4/4			
4.12	**Introduction of Lecture** Queueing Model V (M/M/s:N/FCFS) **Conclusion of Lecture**	L38	5/4			
	**** Class Test/Special lecture/OBT/Quiz**	L39	6/4			
5.1	**UNIT-5:** **DISCRETE PARAMETER MARKOV CHAIN** **Introduction of Lecture** 5.1.1 Introduction 5.1.2 Definition 5.1.3 Markov Chain 5.1.4 Transition Probability **Conclusion of Lecture**	L40	9/4			
	PPT	L41	11/4			
5.2	**Introduction of Lecture** 5.2.1 Steady State Distribution **Conclusion of Lecture**	L42	12/4			

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Introduction of Lecture 5.2.2 Chap Man Kolmogorov Theorem **Conclusion of Lecture**	L43	13/4			
5.3	**Introduction of Lecture** 5.3.1 Classification of States of Markov Chain 5.3.2 Related Question **Conclusion of Lecture**	L44	16/4			
5.4	**Introduction of Lecture** 5.4.1 Queueing Model (M/G/1:∞/GD) **Conclusion of Lecture**	L45	18/4			
5.5	**Introduction of Lecture** 5.5.1 Discrete Parameter Birth Death Process **Conclusion of Lecture**	L46,L47	19/4, 20/4			
	** Class Test/Special lecture/OBT/Quiz	L48 L49	23/4 25/4			

L50 26/4

L51 27/4

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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SYLLABUS DEPLOYMENT

Campus: PCE Course: B.Tech. Class/Section: II Yr Date: 03/01/2018 (A)

N e of Faculty: Dr. Shilpi Jain Name of Subject: SPT Code: 4CS03A

.No.	Topic As Per Blownup Syllabus	Lect. No.	Planned Date	Actual Del. Date	Reason For Deviation	Ref. / Text Book With Page No.
0	**Zero Lecture** **UNIT 1:** **DISCRETE RANDOM VARIABLES** **Introduction of Lecture**	L0				
1.1	1.1.1Introduction to probability 1.1.2 Deterministic experiments 1.1.3 Random Experiments 1.1.4 Trial 1.1.5Definition of sample space 1.1.6 exhaustive events 1.1.7 Mutually exhaustive events **Conclusion of Lecture**	L1				
1.2	**Introduction of Lecture** 1.2.1 Union intersection & complement of events 1.2.2 laws o f events 1.2.3 Probability 1.2.4 Conditional Probability	L2				

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Conclusion of Lecture					
1.3	**Introduction of Lecture** 1.3.1 Theorem of Total probability 1.3.2 Statement & proof of Baye's Theorem **Conclusion of Lecture**	L3				
1.4	**Introduction of Lecture** 1.4.1 Introduction 1.4.2 Definition 1.4.3 probability Mass Function 1.4.4 Discrete & continuous random variables **Conclusion of Lecture**	L4				
1.5	**Introduction of Lecture** 1.5.1 Probability generating Function 1.5.2 Bivariate Random variables **Conclusion of Lecture**	L5				
1.6	**Introduction of Lecture** 1.6.1 Moments 1.6.2 Mathematical Expectation 1.6.3 Computation of Mean time to Failure **Conclusion of Lecture**	L6				
1.7	**Introduction of Lecture** 1.7.1 Introduction 1.7.2 Definition of random process 1.7.3 classification of random process **Conclusion of Lecture**	L7				
1.8	**Introduction of Lecture** 1.8.1 Markov process 1.8.2 Mean 1.8.3 Variance **Conclusion of Lecture**	L8				
1.9	**Introduction of Lecture** 1.9.1 Concept 1.9.2 Poisson counting process	L9				

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	1.9.3 Homogenous Poisson process 1.9.4 Inter arrival time of Poisson process **Conclusion of Lecture**					
1.10	**Introduction of Lecture** 1.10.1 Probability Related Question **Conclusion of Lecture** **** Class Test/Special lecture/OBT/Quiz**	L10				
2.1	**UNIT 2:** **DISCRETE & CONTINUOUS DISTRIBUTION** **Introduction of Lecture** Binomial Distribution 2.1.1 Introduction 2.1.2 Definition 2.1.3 p.m.f 2.1.4 mean 2.1.5 Variance 2.1.6 Moment Generating function **Conclusion of Lecture**	L11				
2.2	**Introduction of Lecture** 2.2.1 Poisson Distribution 2.2.2 Definition 2.2.3 p.m.f. 2.2.4 mean **Conclusion of Lecture**	L12				
2	**Introduction of Lecture** Poisson Distribution 2.3.1 Variance 2.3.2 Moment Generating function **Conclusion of Lecture**	L13				
2.4	**Introduction of Lecture** Normal Distribution 2.4.1 introduction 2.4.2 Definition 2.4.3 p.d.f 2.4.4 mean	L14				

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Conclusion of Lecture					
2.5	**Introduction of Lecture** Normal Distribution 2.5.1 Variance 2.5.2 Moment Generating function **Conclusion of Lecture**	L15				
2.6	**Introduction of Lecture** Rectangular Distribution 2.6.1 Introduction 2.6.2 Definition 2.6.3 p.d.f 2.6.4 mean 2.6..5 Variance **Conclusion of Lecture**	L16				
2.7	**Introduction of Lecture** Rectangular Distribution 2.7.1 Moment Generating function 2.7.2 Related Question **Conclusion of Lecture**	L17				
2.8	**Introduction of Lecture** Exponential distribution 2 .8.1 Introduction 2.8.2 Definition 2.8.3 p.d.f 2.8.4 mean 2.8..5 Variance 2.8.6. Moment Generating function **Conclusion of Lecture**	L18				
	** Class Test/Special lecture/OBT/Quiz	L19				
3.1	**UNIT-3:** **CORRELATION & REGRESSION** **Introduction of Lecture** Correlation 3.1.1 Introduction	L20				

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	3.1.2 Karl Pearson Coefficient 3.1.3 Rank Correlation **Conclusion of Lecture**					
3.2	**Introduction of Lecture** 3.2.1 Curve Fitting 3.2.2 Concept of Method (Least Square Method) 3.2.3 Fitting a Straight line **Conclusion of Lecture**	L21				
3.3	**Introduction of Lecture** 3.3.1 Fitting of a Parabola 3.3.2 Related Question 3.3.3 Fitting of Other Curves **Conclusion of Lecture**	L22				
3.4	**Introduction of Lecture** Regression 3.4.1 Linear regression 3.4.2 Introduction 3.4.3 Definition 3.4.4Regression Equation & Some Theorems **Conclusion of Lecture**	L23				
3.5	**Introduction of Lecture** 3.5.1 Standard Error of Estimate or Residual Variance 3.5.2 Coefficient of Determination 3.5.3 Normal Correlation Analysis **Conclusion of Lecture**	L24				
3.6	**Introduction of Lecture** 3.6.1 Normal Regression Analysis 3.6.2 Related Question **Conclusion of Lecture** **** Class Test/Special lecture/OBT/Quiz** **UNIT 4:** **QUEUEING THEORY**	L25				

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

4.1	**Introduction of Lecture** 4.1.1 Introduction 4.1.2 Queueing system 4.1.3 FIFO & LIFO 4.1.4 Probability distribution of arrival **Conclusion of Lecture**	L26				
4.2	**Introduction of Lecture** 4.2.1 Probability distribution of Waiting time 4.2.2 Birth Death Process (Concept) 4.2.3 Notations **Conclusion of Lecture**	L27				
4.3	**Introduction of Lecture** 4.3.1 Pure Birth Process 4.3.2 Markovian property of Inter arrival time **Conclusion of Lecture**	L28				
4.4	**Introduction of Lecture** 4.4.1 Pure Death Process **Conclusion of Lecture**	L29				
4.5	**Introduction of Lecture** 4.5.1 Birth & Death Process 4.5.2 Kendall's Notation **Conclusion of Lecture**	L30				
4.6	**Introduction of Lecture** 4.6.1 Queueing Model I (M/M/1:∞/FIFO) 4.6.2 Service time Distribution **Conclusion of Lecture**	L31				
4.7	**Introduction of Lecture** Related Problem to Model -I **Conclusion of Lecture**	L32				
	**** Class Test/Special lecture/OBT/Quiz**	L33				
4.8	**Introduction of Lecture**					

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Queueing model II (M/M/1:N/FIFO) **Conclusion of Lecture**	L34				
4.9	**Introduction of Lecture** Related Problem to Model II **Conclusion of Lecture**	L35				
4.10	**Introduction of Lecture** Queueing Model III Generalisation of Model I (M/M/1:∞/FCFS) **Conclusion of Lecture**	L36				
4.11	**Introduction of Lecture** Queueing Model IV (M/M/s:∞/FCFS) **Conclusion of Lecture**	L37				
4.12	**Introduction of Lecture** Queueing Model V (M/M/s:N/FCFS) **Conclusion of Lecture**	L38				
	**** Class Test/Special lecture/OBT/Quiz**	L39				
5.1	**UNIT-5:** **DISCRETE PARAMETER MARKOV CHAIN** **Introduction of Lecture** 5.1.1 Introduction 5.1.2 Definition 5.1.3 Markov Chain 5.1.4 Transition Probability **Conclusion of Lecture**	L40				
	PPT	L41				
5.2	**Introduction of Lecture** 5.2.1 Steady State Distribution **Conclusion of Lecture**	L42				

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

	Introduction of Lecture 5.2.2 Chap Man Kolmogorov Theorem **Conclusion of Lecture**	L43				
5.3	**Introduction of Lecture** 5.3.1 Classification of States of Markov Chain 5.3.2 Related Question **Conclusion of Lecture**	L44				
5.4	**Introduction of Lecture** 5.4.1 Queueing Model (M/G/1:∞/GD) **Conclusion of Lecture**	L45				
.5	**Introduction of Lecture** 5.5.1 Discrete Parameter Birth Death Process **Conclusion of Lecture**	L46,L 47				
	** Class Test/Special lecture/OBT/Quiz	L48				

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

4E 4162

Roll No. ____________________ [Total No. of Pages : 4

4E 4162

B.Tech. IV Semester (Main/Back) Examination, June/July - 2015

Computer Science and Engineering

4CS3A Statistics and Probability Theory

(Common with IT)

Time : 3 Hours

Maximum Marks : 80
Min. Passing Marks : 26

Instructions to Candidates:

*Attempt any **five** questions, selecting **one** question from **each unit**. All questions carry **equal** marks. (Schematic diagrams must be shown wherever necessary. Any data you feel missing suitably be assumed and stated clearly. Units of quantities used/calculated must be stated clearly.*

Use of following supporting material is permitted during examination.

1. Normal distribution table.

Unit - I

1. a) Two computers A and B are to be marketed. A salesman who is assigned the job of finding customers for them has 60% and 40% chances respectively of succeeding in case of computers A and B. The two computers can be sold independently. Given that he was able to sell at least one computer, what is the probability that computer, A has been sold. **(8)**

   b) There are three boxes containing respectively 1 white, 2 red and 3 black balls, 2 white, 3 red and 1 black ball, 3 white, 1 red and 2 black balls. A box is chosen and from it two balls are drawn at random. They happen to be one red and one white. find the probabilities that these come from

   i) The first box

   ii) The second box and

   iii) The third box **(8)**

OR

1. a) The joint probability mass function of (X,Y) is given by $f_{XY}(x, y) = k(2x+3y)$; $x=0,1,2$; $y=1,2,3$ find:

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

i) The value of constant K,

ii) Marginal probability distribution of X and Y

iii) $P\left(\frac{Y=y}{X=2}\right)$ (6)

b) If f(t) be the pdf of a system and h(t) be the hazard rate function of the same system, then using $f(t)=\lambda^2 te^{-\lambda t}$, find h(t) and MTTF. (6)

c) The first four moments of a distribution about the value '5' of the variable are 2,20,40 and 50. Find the mean, variance, β_1 *and* β_2 (4)

Unit - II

2. a) The probability of a man hitting a target is 1/4. Then find:

i) If he fires 7 times, what is the probability of his hitting the target atleast twice

ii) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3 (4+4=8)

b) For Poisson distribution find first four moments about origin and hence find first four central moments (8)

OR

2. a) The average monthly sales of 5000 firms are normally distributed. Its mean and standard deviation are Rs 36000 and Rs 10,000 respectively then find:

i) The number of firms the sales of which are over Rs. 40,000

ii) The number of firms the sales of which are between Rs. 38,500 and Rs.41000 (4+4=8)

b) Subway trains from karolbagh to chandani chowk run every half an hour between midnight and six in the morning what is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes. (4)

c) State and derive memory less property of exponential distribution (4)

Unit - III

3. a) Calculate the karl person's coefficient of correlation between x and y from the following data:

x :	25	27	30	35	33	28	36
y :	19	22	27	28	30	23	28

(8)

b) In a partially destroyed laboratory, record of an analysis of correlation data, the following results only are legible: variance of x=9.

Regression equations: 8x-10y+66=0,

40x-18y=214 then find

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

i) The mean values of x and y

ii) Coefficient of correlation between x and y

iii) The standard deviation of y (8)

OR

3\. a) Use method of least squares to fit a straight line to the following data treating y as the dependent variable

x :	1	2	3	4	5
y :	5	7	9	10	11

(8)

b) Ten competitors in a beauty contest are ranked by three judges in the following order: (8)

I judge :	1	6	5	10	3	2	4	9	7	8
II judge :	3	5	8	4	7	10	2	1	6	9
III judge :	6	4	9	8	1	2	3	10	5	7

Then find that which two judges have better correlation.

Unit - IV

4\. a) If for a period of 2 hours in a day(7 A.M to 9. A.M) customers arrive in a barber's shop that has space to accommodate only 4 customers. Arrival rate of customers is 3 per hour and service time is 36 minutes per customer. Then, find:

i) The probability that there is no customer in the shop and

ii) Average number of customers in the shop (8)

b) In a shop there are two computers for carrying out the job work. the average time per job on each computer is 20 minutes per job and the average arrival rate is 2 jobs per hour. Assume the job times to bedistributed exponentially. If the maximum number of jobs accepted on a day be 6, then find:

i) The expected number of jobs waiting for computer

ii) The total time lost per day consists of 8 working hours (8)

OR

4\. a) On a telephone booth, arrivals of customers follow the Poisson process with an average time of 10 minutes between one arrival and next arrival. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes. Then find: (8)

i) Average number of customers present in the system

ii) The probability that a customer spends more than 10 minutes in the booth.

iii) The fraction of a day that the phone will be in use

b) A supermarket has two girls serving at the counters. The customers arrive in a Poisson fashion at the rate of 12 per hour. The service time for each customer is exponential with mean 6 minutes. Then find:

i) The probability that an arriving customer has to wait for service

ii) The average number of customers in the system

iii) The average time spent by a customer in the supermarket (8)

Unit - V

5. a) A housewife buys three kinds of food A,B and C. She never buy the same food on successive weeks. If she buys food A, then the next week she buys food B. However if she buys B or C. Then the next week she is three times as likely to buy A as to the other brand. Find the transition probability matrix (8)

b) An automata car station has one bay where service is done. The arrival pattern is Poisson with 4 cars per hour and may wait in the parking lot in the street if the bay is busy then find the time spent in the station by a car if service time distribution is normal with mean 12 minutes and $\sigma = 3$ minutes. Also, find the average number of cars in the station, if service - time distribution is uniform between 8 and 20 minutes (8)

OR

5. a) Write a short note on 'discrete parameter birth - death process' (8)

b) In a heavy machine shop the overhead crane is utilized 75%. Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of 8.8 minutes. What is the average calling time for the services of the crane and what is the average delay in getting service? If the average service time is cut to 8 minutes with standard deviation of 6 minutes, how much reduction will occur on average in the delay of getting served? (8)

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
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Roll No. ____________

Total No of Pages: 4

4E4162

4E4162
B. Tech. IV Sem. (Main/Back) Exam., June/July-2014
Computer Science and Engineering
4CS3A Statistics and Probability Theory
Common with IT

Time: 3 Hours

Maximum Marks: 80
Min. Passing Marks: 24

Instructions to Candidates:-

*Attempt any **five questions**, selecting **one question** from **each unit**. All Questions carry **equal** marks. Schematic diagrams must be shown wherever necessary. Any data you feel missing suitably be assumed and stated clearly.*
Units of quantities used/ calculated must be stated clearly.

Use of following supporting material is permitted during examination.

(Mentioned in form No.205)

1.____________________ 2.____________________

UNIT-I

Q.1 (a) Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots. [8]

(b) In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5%, 4% and 2% are respectively defective bolts. A bolt is drawn at random from the product, and is found to be defective, What is the probability that it is manufactured by machine A, B and C? [8]

OR

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Q.1 (a) Given the joint probability density [8]

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y), 0 < x < 1, 0 < y < 1 \\ 0 \quad : \quad \text{Otherwise} \end{cases}$$

Find.

(i) Marginal density of X and Y.

(ii) Conditional density of X given Y = y and use it to evaluate $P\left\{\frac{X \le 1/2}{Y = 1/2}\right\}$

(b) Let f(t) be the pdf of time to failure T of a system and h(t) be the hazard rate function. Find h(t) and MTTF when. $f(t) = \lambda^2 t e^{-\lambda t}$ [8]

UNIT-II

Q.2 (a) Out of 800 families with 4 Children each, How many families would be expected to have [8]

(i) 2 boys and 2 girls

(ii) at least 1 boy

(iii) at most and girls. Assume equal probabilities for boys and girls

(b) Fit a Poisson distribution to the following data which gives the number of dodders in a sample of clover seeds [8]

No. of Dodders (x):	0	1	2	3	4	5	6	7	8
Observed frequency (f):	56	156	132	92	37	22	4	0	1

OR

Q.2 (a) Find the mean and Variance of Poisson Distribution. [8]

(b) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the parameters of distribution. (Given ϕ (0.50) = 0.19 and ϕ (1.41) = 0.42) [8]

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

UNIT-III

Q.3 (a) Calculate the karl Pearson's Coefficient of correlation of the following data: [8]

X	:	25	27	30	35	33	28	36
Y	:	19	22	27	28	30	23	28

(b) Show that the angle between the lines of regression is given by: [8]

$$\text{Tan } \theta = \pm \left(\frac{1-r^2}{r}\right)\frac{\sigma_x \sigma_y}{(\sigma_x^{\ 2} + \sigma_y^{\ 2})}$$

OR

Q.3 (a) Obtain the rank correlation Coefficient for the following data: [8]

x :	68	64	75	50	64	80	75	40	55	64
y :	62	58	68	45	81	60	68	48	50	74

(b) Lines 2x+3y = 10 and 4x+5y = 18 are lines of regression between two variables x and y. Decide which one is the line of regression of x on y. Given x =5, find y and also find mean values of Variables. [8]

UNIT-IV

Q.4 (a) Write short note on Pure Birth death process. [8]

(b) Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially, with mean 3 minutes. Find: [8]

(i) What is the probability that a person arriving at the booth will have to wait?

(ii) What is the average length of queue that form from time to time?

(iii) The telephone company will install a second booth when convinced that an arrival would have to wait at least 3 minutes for the phone. By how much must the flow of arrivals be increased in order to justify for a second booth?

OR

Q.4 (a) If for a period of 2 hours in a day (8-10 AM), trains arrive at the yard (Capacity of which is 4 trains) in every 20 minutes, but the service time remains 36 minutes. Then calculate for this period: [8]

(i) The probability that the yard is empty.

(ii) The average queue length.

(b) A Supermarket has two girls serving at the counters. The Customers arrive in a Poisson fashion at the rate of 12 per hour. The service time for each customer is exponential with mean 6 minutes. Find: [8]

(i) The probability that an arriving customer has to wait for service.

(ii) The average number of customers in the system.

(iii) The average time spent by a customer in the supermarket.

UNIT-V

Q.5 (a) Write short notes on the following: [8]

(i) Discrete parameter Markov chain.

(ii) Transition probability Matrix

(b) Corresponding to a Markov chain, the initial provability matrix $P^{(o)} = \left(\frac{1}{4}, \frac{3}{4}\right)$

and transition probability matrix (tpm) is $P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$ where A and B

denote the two states of the process. Find:

(i) The probability of reaching state A after two steps $P_A(2)$.

(ii) The probability of state B after two steps.

(iii) $[t_{ij}]$ matrix when $n \to \infty$ [8]

OR

Q.5 (a) Describe briefly the (M/G/1) : (∞/GD) queuing system analysing the steady State solution. [8]

(b) In a heavy machine shop, the overhead crane is 75% utilized. Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of 8.8 minutes. What is the average calling time for the service of the crane and what is the average delay in getting service?

If the average service time is cut to 8 minutes, with standard deviation of 6.0 minutes, how much reduction will occur on an average in the delay of getting served? [8]

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B. Tech. IV-Sem. (Main & Back) Exam; April-May 2017

Computer Sci. & Engg.

4CS3A Statistics & Probability Theory

CS, IT

Time : 3 Hours **Maximum Marks : 8**

Min. Passing Marks : 2

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Use of following supporting materials is permitted during examination. (Mentioned in form No. 205)

1. Normal distribution - Table   2. NIL

UNIT - I

1 (a) The probability that a teacher will give an unannounced test during any class meeting is 1/5. If a student is absent twice, what is the probability that he will miss at least one test ? **8**

(b) The first four moments of a distribution about the value 5 of the variate are 2, 20, 40 and 50. Also find mean and variance of the distribution. **8**

OR

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

1 (a) Two random variables X and Y have the following joint probability densit function :

$$f(x,y)=\begin{cases}2-x-y, & 0\le x\le 1, 0\le y\le 1\\ 0 & \text{, otherwise}\end{cases}$$

Find

(i) Marginal probability density functions of x and y

(ii) Conditional density functions

(iii) Var (X) and Var (Y)

(b) If the life time of a component has probability density function $\lambda e^{-\lambda t}$, $t>0$

Compute its time to failure and variance.

Also define the mean time to failure in terms of the reliability function

8

UNIT - II

2 (a) Determine the mean and variance of binomial distribution. Also define moment generating function of binomial distribution.

8

(b) A driver has two taxies, which he hires out day by day. The number of demands for a taxi on each day is distributed as a Poisson variate with mean 1.5. Calculate the proportion of days on which

(i) neither of the cars is used

(ii) some demand is refused (Given $e^{-1.5} = 0.2231$).

8

OR

2 (a) As a result of tests on 20,000 electric bulbs manufactured by a company it was found that the life time of the bulb was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. On the basis of the information estimate the number of the bulbs that is expected to burn for (i) more than 2150 hours (ii) less than 1960 hours.

8

(b) Define exponential distribution . Show that for the exponential distribution given by $dp = ae^{-\frac{x}{c}}$, $0 \le x < \infty$, $c > 0$ a being a constant, the mean and the standard deviation are each equal to C.

8

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

UNIT - III

3 (a) Calculate the coefficient of correlation between x and y using the following data :

x:	1	2	3	4	5	6	7	8	9
y:	9	8	10	12	11	13	14	16	15

8

(b) Calculate rank correlation coefficient for the following data :

x:	81	78	73	73	69	68	62	58
y:	10	12	18	18	18	22	20	24

8

OR

3 (a) Write a short note on linear regression and obtain the regression line of y on x.

4+4=8

(b) Fit a second degree parabola to the following data :

x:	0	1	2	3	4
y:	1	5	10	22	38

8

UNIT - IV

4 (a) On a telephone booth, arrivals of customers follow the Poisson proce: with an average time of 10 minutes between one arrival and next arriva The length of a phone call is assumed to be distributed exponentially wit mean 3 minutes.

(i) Find the average number of persons waiting in the system.

(ii) What is probability that a customer spends more than 10 minute in the booth ?

(iii) Find the fraction of a day when the phone will be used.

(b) Assume that the trucks with goods are coming in a market yard at th rate of 30 trucks per day and suppose that the inter-arrival times follov an exponential distribution. The time to unload the trucks is assumed t be exponential with an average of 42 minutes. If the market yard can admi 10 trucks at a time, calculate P (the yard is empty) and find the averag length of queue.

8

OR

4 (a) Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room cannot accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(i) Find the effective arrival rate at the clinic.

(ii) What is the probability that an arrival patient will not wait ?

(iii) What is the expected waiting time until a patient is discharged from the clinic ?

8

(b) A super market has two girls serving at the counters. The customers arrive in a Poisson fashion at the rate of 12 per hour. The service time for each customer is exponential with mean 6 minutes. Find :

(i) the probability that an arriving customer has to wait for service.

(ii) the average number of customers in the system.

(iii) the average time spent by a customer in the super market.

8

UNIT - V

5 (a) Write a short note on discrete parameter Markov chain.

8

(b) Two brands A and B of a product have probabilities 30% and 70% respectively at time t = 0, if their transition matrix P be $\begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$, find their probabilities

(i) after time t = 1,

(ii) after time t = 2

(iii) their steady state probabilities.

8

OR

5 (a) Automata car wash facility operates with only one bay. Cars arrive according to Poisson distribution, with a mean of 4 cars per hour and may wait in the facilities parking lot if the bay is busy. Find the time spent by a car in the system and in the waiting if

(i) the time for washing and cleaning a car is exponential with a mean of 10 minutes

(ii) the time of washing and cleaning a car is constant and is equal to 10 minutes. Which facility is better ?

8

(b) Write a short note on M/G/1 queuing model.

8

Solution

I-Mid term paper (4CS3A)

SPT.

Submitted by
(Dr. Shilpi Jain)

Sol^n 1 (a) $P(\text{defective}) = P(D) = P(D/A)P(A) + P(D/B)P(B) + P(D/C)P(C)$
$= 0.069$

By Baye's Th.

$$P(A/D) = \frac{P(A)\cdot P(D/A)}{P(D)} = \frac{0.024}{0.069} \simeq 0.348$$

$$P(B/D) = \frac{0.025}{0.069} \simeq 0.362$$

$$P(C/D) = \frac{0.02}{0.069} \simeq 0.29$$

Sol^n 1 (b) $P(\text{getting sum of numbers on two dice as } 9) = \frac{4}{36} = \frac{1}{9} = p$

$P(\text{not getting " " }) = 1 - p = \frac{8}{9} = q$

A is the first to throw

$\therefore P(\text{A winning}) = p + q^2p + q^4p + \cdots = \frac{1/9}{1-\frac{64}{81}} = \frac{9}{17}$

$P(\text{B winning}) = qp + q^3p + \cdots = \frac{8}{17}$

$\therefore P\left(\frac{A}{B}\right) = 9 : 8.$

Sol^n 2(a)

Statement of Bayes Th: If $B_1, B_2, \ldots B_n$ be a ~~part~~ form of partition of sample ~~sapace~~ space S & $P(B_i) \neq 0$ for $i = 1, 2, \ldots n$ and A be any event in S then

$$P\left(\frac{B_i}{A}\right) = \frac{P\left(\frac{A}{B_i}\right)P(B_i)}{\sum^{n} P\left(\frac{A}{B_k}\right)P(B_k)}$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Proof. Here $B_1 \cup B_2 \cup \cdots \cup B_n = S$

& $B_i \cap B_j = \phi$ for $i \neq j$

$$P(A \cap B_i) = P\left(\frac{A}{B_i}\right) P(B_i) = P\left(\frac{B_i}{A}\right) P(A)$$

$$\Rightarrow P\left(\frac{B_i}{A}\right) = \frac{P\left(\frac{A}{B_i}\right) P(B_i)}{P(A)} = \frac{P\left(\frac{A}{B_i}\right) P(B_i)}{\sum_{k=1}^{n} P\left(\frac{A}{B_k}\right) P(B_k)}$$ {by Th. of total prob.}

sol^n 2(b) Here $\mu_1'' = -4$, $\mu_2'' = 22$, $\mu_3'' = -117$, $\mu_4'' = 560$

$a = 5$

$$\mu_1'' = \bar{x} - a \Rightarrow \boxed{\bar{x} = 1 = \mu_1'}$$

First four moment about mean

$$\boxed{\mu_1 = 0}, \quad \boxed{\mu_2 = \mu_2'' - \mu_1'^2 = 6}$$

$$\mu_3 = \mu_3'' - 3\mu_2''\mu_1' + 2\mu_1'^3$$

$$\Rightarrow \boxed{\mu_3 = 19}$$

$$\boxed{\mu_4 = \mu_4'' - 4\mu_3''\mu_1' + 6\mu_2''\mu_1'^2 - 3\mu_1'^4 = 32}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(19)^2}{(6)^3} > 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{32}{36} < 3$$

~~The curve is~~ positive skewed & platykurtic curve.

sol^n 3(a) Mean of Poisson distribution

$$\mu_1' = \bar{x} = \sum_{x=0}^{\infty} x \frac{e^{-m} m^x}{x!} = m$$

second moment about origin

$$\mu_2' = \sum_{x=0}^{\infty} x^2 \frac{e^{-m} m^x}{x!} = m^2 + m$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Sol^n 3(b) Given

$n = 10, \; p = 0.65, \; q = 1-p = 0.35$

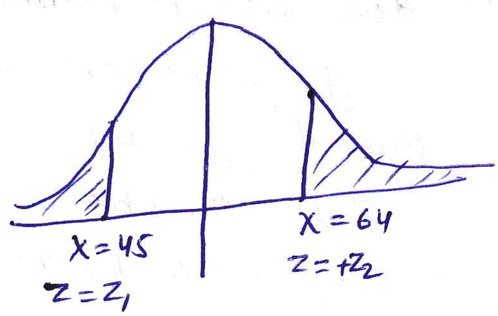
$P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$

$P(X \geq 7) = P(7) + P(8) + P(9) + P(10) = \underline{\underline{0.513}}$

Sol^n 4(a) $P(X<45) = 0.31$ & $P(X>64) = 0.08$

$-z_1 = \dfrac{45-\mu}{\sigma} \longrightarrow (1)$

$z_2 = \dfrac{64-\mu}{\sigma} \longrightarrow (2)$



Now

$P(X<45) = 0.31$

$\Rightarrow \boxed{z_1 = 0.50}$

Also $P(X>64) = 0.08$

$\Rightarrow P(Z>z_2) = 0.08$

$\Rightarrow \boxed{z_2 = 1.41}$

from (1) & (2)

$\boxed{\mu \simeq 50, \; \sigma \simeq 10}$

Sol^n. 4(b) Mean of rectangular distribution

$\bar{x} = \mu_1' = \int_a^b x f(x)\, dx$

$= \dfrac{1}{b-a}\left[\dfrac{x^2}{2}\right]_a^b = \dfrac{b^2-a^2}{2(b-a)} = \dfrac{b+a}{2}$

$\Rightarrow \boxed{\mu_1' = \dfrac{b+a}{2}}$

Again $\mu_2' = \dfrac{b^2+ab+a^2}{3}$

Therefore

Variance $\mu_2 = \mu_2' - {\mu_1'}^2 = \left(\dfrac{b^2+ab+a^2}{3}\right) - \dfrac{(b+a)^2}{4} = \dfrac{(b-a)^2}{12}.$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

$\underline{\text{sol}^n\ 5(a)}$ $\bar{x} = \frac{34}{6} = 5.67$, $\bar{y} = \frac{90}{6} = 15$

coeff. of correlation

$$r = \frac{\frac{1}{N}\Sigma xy - \bar{x}\bar{y}}{\sqrt{\left(\frac{1}{N}\Sigma x^2\right) - \bar{x}^2}\sqrt{\left(\frac{1}{N}\Sigma y^2\right) - \bar{y}^2}} = \underline{0.988}$$

$\underline{\text{sol}^n\ 5(b)}$ ~~Here~~ Let the parabola to be fitted be

$$y = a + bx + cx^2 \longrightarrow (1)$$

Normal eq^{n}s are

$$\Sigma y = 7a + b\Sigma x + c\Sigma x^2 \longrightarrow (2)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \longrightarrow (3)$$

$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \longrightarrow (4)$$

Here $\Sigma x = 17.5$, $\Sigma y = 16.8$, $\Sigma x^2 = 50.75$, $\Sigma x^3 = 161.875$

$\Sigma x^4 = 516.9375$, $\Sigma xy = 49.15$, $\Sigma x^2y = 158.225$

from (2), (3) & (4)

$\boxed{a = 1.04}$ $\boxed{b = -0.20}$, $\boxed{c = 0.24}$

from (1)

$$\boxed{y = 1.04 - 0.20x + 0.24x^2}$$

sol^n 6(a). Let the line of regression

y on x be $2x + 3y = 10 \Rightarrow y = -\frac{2}{3}x + \frac{10}{3}$

x on y be $4x + 5y = 18 \Rightarrow x = -\frac{5}{4}y + \frac{18}{4}$

$r^2 = b_{yx} \cdot b_{xy} = \frac{5}{6} = 0.833 < 1$

Which is true. Our assumption is true.

Line of regression x on y is $4x + 5y = 18$

given $x = 5 \Rightarrow y = 0$

$y = 2 \Rightarrow x = 2.$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
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Sol^n 6 (b) Eq^n of line of regression y on x.

$$y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x}) \longrightarrow (1)$$

Eq^n of line of regression x on y

$$x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y}) \longrightarrow (2)$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = m_1$$

$$b_{xy} = \frac{1}{r}\frac{\sigma_x}{\sigma_y} = m_2$$

$$\tan\theta = \frac{m_1-m_2}{1+m_1 m_2} = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

Case I: $r=0 \Rightarrow \tan\theta = \infty \Rightarrow \theta = \pi/2$

lines are $\perp$ to each other.

Case II: $r = \pm 1$, $\theta = 0$

line are coincident.

Sol^n 7(a)

(i) $\sum_{j=1}^{3}\sum_{i=0}^{2} P_{ij} = 1 \Rightarrow 72K = 1 \Rightarrow \boxed{K = \frac{1}{72}}$

(ii) $P(X=0) = P_{0*} = P_{01}+P_{02}+P_{03} = \frac{18}{72}$

$P(X=1) = P_{1*} = \frac{24}{72}$, $P(X=2) = P_{2*} = \frac{30}{72}$

(iii)

$P(Y=1) = P_{*1} = \frac{15}{72}$

$P(Y=2) = P_{*2} = \frac{24}{72}$

$P(Y=3) = P_{*3} = \frac{33}{72}$

(iv) Conditional dis. of X given $Y=1$

$$P\left(\frac{X=i}{Y=1}\right) = \frac{P_{i1}}{P_{*1}} = \frac{P_{i1}}{15K}$$

for ~~$x=0$~~ $P_{..} + P_{..} + P_{..}$ ...

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
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Sol^n 7(b) $E(X) = \sum_i x_i P_i$

x_i	20	18	10
P_i	0.7	0.1	0.2

$$E(X) = 20\times 0.7 + 18\times 0.1 + 10\times 0.2 = 17.80 \text{ Rs.}$$

Sol^n 8(a) Given $f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}, \; t>0$

$$F(t) = \int_0^t f(t)\,dt = 1 - e^{-\alpha t^{\beta}}$$

$$\Rightarrow R(t) = 1 - F(t) = e^{-\alpha t^{\beta}}$$

Now

(i) Failure rate function $\lambda(t) = \frac{f(t)}{R(t)} = \alpha\beta t^{\beta-1}$

(ii) MTTF $= \int_0^{\infty} t\, f(t)\,dt = \alpha^{-1/\beta}\,\Gamma\left(\frac{1}{\beta}+1\right)$

Sol^n 8(b)

(i) $P(\text{second ball is white}) = P(W_2) = P\left(\frac{W_2}{W_1}\right)P(W_1) + P\left(\frac{W_2}{B_1}\right)P(B_1)$

$$= \left(\frac{3}{6}\right)\left(\frac{4}{7}\right) + \left(\frac{4}{6}\right)\left(\frac{3}{7}\right)$$

$$= \frac{12}{42} + \frac{12}{42} = \frac{24}{42} = \frac{4}{7}$$

(ii) $$P\left(\frac{W_1}{W_2}\right) = \frac{P\left(\frac{W_2}{W_1}\right)P(W_1)}{P(W_2)} = \frac{\left(\frac{1}{2}\right)\left(\frac{4}{7}\right)}{4/7} = \frac{1}{2}.$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Solutions.

I - Mid Term Paper

Statistics & Probability Theory (4IT3A). (Dr Shilpi Jain)

Q.1. (a)

	G_1	G_2	G_3
I way	G	B	B
II way	B	G	B
III way	B	B	G

$$P(1G \,\&\, 2B) = P(\text{I way}) + P(\text{II way}) + P(\text{III way})$$

$$= \tfrac{3}{4}\times\tfrac{2}{4}\times\tfrac{3}{4} + \tfrac{1}{4}\times\tfrac{2}{4}\times\tfrac{3}{4} + \tfrac{1}{4}\times\tfrac{2}{4}\times\tfrac{1}{4}$$

$$= \frac{18}{64} + \frac{6}{64} + \frac{2}{64} = \frac{26}{64}$$

$$= \frac{13}{32}$$

(b.) Theorem of total probability :

If events $B_1, B_2, \ldots B_n$ be a form of partition of sample space and $P(B_i) \neq 0$. for $i = 1, 2, \ldots, n$

Let A be any event in S, then

$$\boxed{P(A) = \sum_{i=1}^{n} P(A/B_i)P(B_i)}$$

Proof: If events $B_1, B_2, \ldots B_n$ be a form of partition of sample space S, then

$$B_1 \cup B_2 \cup \ldots \cup B_n = S$$

$$\& \quad B_i \cap B_j = \phi, \quad i \neq j$$

Let A be any part in S,

Now,

$$A = A \cap S$$

$$A = A \cap (B_1 \cup B_2 \cup \ldots \cup B_n)$$

$$A = (A \cap B_1) \cup (A \cap B_2) \ldots \cup (A \cap B_n)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots P(A \cap B_n)$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
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$A \cap B_i$ are disjoint for $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A/B_i)\, P(B_i)$$

· Bayes' Theorem :

If events $B_1, B_2, \ldots B_n$ be a form of partition of sample space and $P(B_i) \neq 0$.

for $i = 1, 2, 3, \ldots, n$

Let A be any event in S,

then,
$$P\left(\frac{B_i}{A}\right) = \frac{P\left(A/B_i\right) P(B_i)}{\sum_{i=1}^{n} P\left(A/B_K\right) P(B_K)}$$

By Theorem of total probability

$$P(A \cap B_i) = P\left(A/B_i\right) P(B_i) = P\left(\frac{B_i}{A}\right) P(A)$$

$$P\left(\frac{B_i}{A}\right) = \frac{P\left(A/B_i\right) P(B_i)}{P(A)}$$

$$\boxed{P\left(\frac{B_i}{A}\right) = \frac{P(A/B_i)\, P(B_i)}{\sum_{i=1}^{n} P\left(A/B_K\right) P(B_K)}}$$

Q.2. (a.) $P(\text{guess}) = 1/3$, $P(\text{copy}) = 1/6$

$$P(\text{skill}) = 1 - P(\text{guess}) - P(\text{copy})$$
$$= 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$$

$$P\left(\frac{\text{corr}}{\text{copy}}\right) = \frac{1}{8}, \quad P\left(\frac{\text{corr}}{\text{skill}}\right) = 1, \quad P\left(\frac{\text{correc}}{\text{guess}}\right) = 1/4$$

$$\therefore P\left(\frac{\text{skill}}{\text{corr}}\right) = \frac{P\left(\frac{\text{corr}}{\text{skill}}\right) P(\text{skill})}{P(\text{skill})\, P\left(\frac{\text{corr}}{\text{skill}}\right) + P\left(\frac{\text{corr}}{\text{copy}}\right) P(\text{copy}) + P\left(\frac{\text{corr}}{\ldots}\right) P(g\ldots}$$

$$P\left(\frac{\text{skill}}{\text{corr}}\right) = \frac{1 \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{8} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{4}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{48} + \frac{1}{12}} = \frac{24}{29}$$

6. Marginal density of X:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

$$= \int_0^1 \frac{2}{3}(x+y)\,dy$$

$$= \frac{2}{3} \int_0^1 (x+y)\,dy$$

$$= \frac{2}{3}\left[xy + \frac{y^2}{2}\right]_0^1$$

$$= \frac{2}{3}\left[x + \frac{1}{2}\right] = \frac{2x}{3} + \frac{1}{3} \quad \begin{cases} \frac{1}{3}(2x+1), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Marginal density of Y:

$$\cancel{f\left(\frac{x}{y}\right) = \frac{f(x,y)}{f_y}} \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx$$

$$= \int_0^1 \frac{2}{3}(x+y)\,dx$$

$$= \frac{2}{3}\left[\frac{x^2}{2} + y\right]_0^1$$

$$= \frac{2}{3}\left[\frac{1}{2} + y\right] = \frac{1}{3} + \frac{2}{3}y$$

$$\begin{cases} \frac{1}{3}[1 + 2y], & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Conditional density of x given $Y = y$

$$f(x/y) = \frac{f(x,y)}{f_y(y)} = \frac{\frac{2}{3}(x+y)}{\frac{1}{3}[1+2y]} = \frac{2(x+y)\cancel{3}}{\cancel{3}(1+2y)}$$

$$\begin{cases} \frac{2(x+y)}{(1+2y)}, & 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$$

Q.3.

a. Mean of Binomial Distribution →

$$\mu_1' = \bar{x} = \sum_{r=0}^{n} p_r x_r$$

$$= \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r} r$$

$$= \sum_{r=0}^{n} \frac{n!\, p^r q^{n-r} r}{r!\,(n-r)!}$$

$$= \sum_{r=0}^{n} \frac{n!\, p^r q^{n-r}}{(r-1)!\,(n-r)!}$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!\, p^{r-1} q^{n-r}}{(r-1)!\,[(n-1)-(r-1)]!}$$

$$= np \sum_{r=1}^{n} {}^{n-1}C_{r-1}\, p^{r-1} q^{n-r}$$

$$= np\,[q^{n-1} + {}^{n-1}C_1\, p q^{n-2} + {}^{n-1}C_{n-2}\, p^2 q^{n-3} + \cdots + p^{n-1}]$$

$$= np\,(q+p)^{n-1}$$

$$= np$$

- Second Moment about origin:

$$\mu_2' = \sum_{r=0}^{n} p_r x_r^2$$

$$= \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r} r^2$$

$$= \sum_{r=0}^{n} \frac{n!\, p^r q^{n-r} r^2}{r!\,(n-r)!}$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

$$= \sum_{r=1}^{n} \frac{n!\, p^r q^{n-r}}{(r-1)!\,(n-r)!}$$

$$= \sum_{r=2}^{n} \frac{n!\, p^r q^{n-r}}{(r-2)!\,(n-r)!} + \sum_{r=1}^{n} \frac{n!\, p^r q^{n-r}}{(r-1)!\,(n-r)!}$$

$$= n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!\, p^{r-2} q^{n-r}}{(r-2)!\,((n-2)-(r-2))!} + np$$

$$= n(n-1)p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2}\, p^{r-2} q^{n-r} + np$$

$$= n(n-1)p^2 \left[q^{n-2} + {}^{n-2}C_1\, p q^{n-3} + {}^{n-2}C_2\, p^2 q^{n-4} + \cdots + p^{n-2} \right] + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$\boxed{\mu_2' = n(n-1)p^2 + np}$$

- Variance of Binomial Distribution:

$$\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2$$

$$= \cancel{n^2}p^2 - np^2 + np - \cancel{n^2}p^2$$

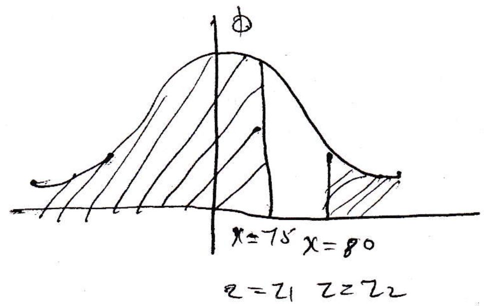
$$= np(1-p) = npq$$

$$\sigma = \sqrt{npq}$$

b.

$$P(X < 75) = 0.58 \quad \text{--- (1)}$$

$$P(X > 80) = 0.04 \quad \text{--- (2)}$$



$\therefore$ when $x = 75$, $z_1 = \dfrac{75-\mu}{\sigma}$ --- (3)

$x = 80$, $z_2 = \dfrac{80-\mu}{\sigma}$ --- (4)

...ce, $P(X < 75) = 0.58$

$P(z < z_1) = 0.58$

$0.5 + P(0 < z < z_1) = 0.58$

$P(0 < Z < z_1) = 0.08$

$z_1 = 0.20$

and $P(X > 80) = 0.04$

$P(z > z_2) = 0.04$

$0.5 - P(0 < Z < z_2) = 0.04$

$P(0 < Z < z_2) = 0.46$

$z_2 = 1.75$

On solving eq^n ③ & ④

$0.20\sigma = 75 - \mu$ —— ⑤

$1.75\sigma = 80 - \mu$ —— ⑥

Divide eq^n ⑤ & ⑥

$\boxed{\mu = 74.4}$

$\therefore$ $\boxed{\sigma = 3.5}$

Q4. a) X = Number of defective bulbs

$p = 0.01$

$m = np$

$= 10 \times 0.01$

$= 0.1$

$P(X = r) = \dfrac{e^{-m} m^r}{r!}$

i. $P(X = 0) = \dfrac{e^{-0.1} (0.1)^0}{0!} = 0.9048$

Req. No of Packets $= 10000 \times 0.9048$

$= 9048$ packets

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

(ii.) $P(X=1) = \dfrac{e^{-0.1}(+0.1)^1}{1!} = 0.09048$

Required no. of packets $= 10000 \times 0.09048$

$= 904.8$ packets

(iii.) $P(X=2) = \dfrac{e^{-0.1}(0.1)^2}{2!} = 0.004524$

Required no. of packets $= 0.004524 \times 10000$

$= 45.24 \approx 45$ packets.

b. Mean of Rectangular Distribution:

$M_x(t) = E(e^{tx})$

$= \int_a^b f(x) e^{tx} dx$

$= \dfrac{1}{b-a}\left[\dfrac{e^{tx}}{t}\right]_a^b$

$= \dfrac{1}{b-a}\left[\dfrac{e^{tb} - e^{ta}}{t}\right]$

$= \dfrac{1}{(b-a)t}\left[\left(1+bt+\dfrac{(bt)^2}{2!}+\dots\right) - \left(1+at+\dfrac{(at)^2}{2!}+\dots\right)\right]$

$= \dfrac{1}{(b-a)t}\left[t(b-a) + \dfrac{(b^2-a^2)t}{2!} + \dfrac{(b^2-a^2)t^2}{(b-a)\,2!} + \dots\right]$ — ①

We know that

$M_x(t) = 1 + \mu_1' t + \mu_2' \dfrac{t^2}{2!} + \dots$ — ②

compare the coeff. of t in eq^n 1 & 2

$\mu_1' = \dfrac{a+b}{2} = $ Mean

$\mu_2' = \dfrac{b^3-a^3}{(b-a)3} = \dfrac{\cancel{(b-a)}(b^2+a^2+ab)}{\cancel{(b-a)}3} \Rightarrow$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

...iance of Rectangular Distribution :

$$\sigma^2 \text{ or } \mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{b^2+a^2+ab}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4b^2+4a^2+4ab-3b^2-3a^2-6ab}{12}$$

$$= \frac{b^2+a^2+ab}{12} = \frac{(b-a)^2}{12}$$

Q.5. a.) $n=18$, $\Sigma x^2 = 60$, $\Sigma y^2 = 96$, $\Sigma x = 12$, $\Sigma y = 18$, $\Sigma xy = 48$.

$$\bar{x} = \frac{\Sigma x}{18} = \frac{12}{18} = 0.666$$

$$\bar{y} = \frac{12}{12} = 1$$

$$r = \frac{\frac{1}{N}\Sigma xy - \bar{x}\bar{y}}{\sqrt{\left(\frac{1}{N}\Sigma x^2\right) - \bar{x}^2}\sqrt{\left(\frac{1}{N}\Sigma y^2\right) - \bar{y}^2}}$$

$$r = \frac{\frac{1}{18}(48) - 0.666}{\sqrt{\frac{1}{18}(60) - 0.4435}\sqrt{\frac{1}{18}(96) - 1}}$$

$$r = \frac{2}{\sqrt{2.8895}\sqrt{4.333}}$$

$$r = \frac{2}{3.535}$$

$$\boxed{r = 0.565}$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Q.

x	1	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.6	2.7	3.4	4.1

The eq^n is

$$y = a + bx + cx^2$$

Then the normal eq^n are

$$\Sigma y = 7a + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

x	y	x^2	x^3	x^4	xy	x^2y
1	1.1	1	1	1	1.1	1.1
1.5	1.3	2.25	3.375	5.0625	1.95	2.925
2.0	1.6	4	8	16	3.2	6.4
2.5	2.6	6.25	15.625	7.8125	6.5	16.25
3.0	2.7	9	27	81	8.1	24.3
3.5	3.4	12.25	42.875	150.0625	11.9	41.65
4.0	4.1	16	64	256	16.4	65.6
17.5	16.8	50.75	161.875	516.9375	49.15	158.225

Subsituting these values in normal eq^n,

$$16.8 = 7a + 17.5b + 50.75c$$

$$49.15 = 17.5a + 50.75b + 161.875c$$

$$158.225 = 50.75a + 161.875b + 516.9375c$$

Solving these we get,

$$a = 1.04, \quad b = -0.20, \quad c = 0.24$$

Hence, the required parabola is

$$y = 1.04 - 0.20x + 0.24x^2$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

a.)

$$m_1 = \frac{1}{r}\frac{\sigma y}{\sigma x}$$

$$m_2 = \frac{r \sigma y}{\sigma x}$$

$$\therefore \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta = \frac{\frac{1}{r}\frac{\sigma y}{\sigma x} - \frac{r\sigma y}{\sigma x}}{1 + \frac{1}{r}\frac{\sigma y}{\sigma x}\frac{r\sigma y}{\sigma x}}$$

$$\tan\theta = \frac{\frac{(1-r^2)}{r}\ \frac{\sigma y}{\sigma x}\ \sigma x^2}{(\sigma x^2 + \sigma y^2)}$$

$$\boxed{\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma x \sigma y}{\sigma x^2 + \sigma y^2}\right)}$$

Case 1: If $r = 1 \Rightarrow \theta = 0$

Therefore, the lines are coincide.

Case 2: If $r = 0 \Rightarrow \theta = \pi/2$

Therefore, lines are perpendicular to each other.

b.)

X	Y	Rank in X	Rank in Y	$d_i = x_i - y_i$	d_i^2
68	62	4	5	-1	1
64	58	6	7	-1	1
75	68	2.5	3.5	-1	1
50	45	9	10	-1	1
64	81	6	1	5	25
80	60	1	6	-5	25

75	68	2.5	3.5	-1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	74	6	2	4	16
				0	72

correlation Factor for X where 2.5 is repeated two times.

$$\frac{2(2^2-1)}{12} = \frac{1}{2}$$

when 6 is repeated 3 times

$$\frac{3(3^2-1)}{12} = 2$$

Correlation Factor for Y when 3.5 is repeated 2 times

$$\frac{2(2^2-1)}{2} = \frac{1}{2}$$

$$\therefore \quad r = 1 - \frac{6(\Sigma d^2 + C.F.)}{n(n^2-1)}$$

$$r = 1 - \frac{6[72 + 1/2 + 2 + 1/2]}{10(10^2-1)}$$

$$r = 1 - \frac{6 \times 75}{10 \times 99} \Rightarrow 1 - \frac{5}{11} = \frac{6}{11}$$

$$\boxed{r = 0.545}$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

i)

Variable	0 - 10	10 - 20	20 - 30	30 - 40
Frequency	1	3	4	2

$$\bar{x} = \mu_1' = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{220}{10}$$

$$\boxed{\bar{x} = 22}$$

x	f	$f.x$	$(x-22)$	$f(x-22)$	$(x-22)^2$	$f(x-22)^2$	$(x-22)^3$	$f(x-22)^3$	$(x-22)^4$	$f(x-22)^4$
5	1	5	-17	-17	289	289	4913	-4923	83521	83521
15	3	45	7	-21	49	147	343	1029	2401	7293
25	4	100	3	12	9	36	27	108	81	324
35	2	70	13	26	169	338	2197	4394	28561	57122
	10	220		0		810		-1440		148170

$$\mu_1 = 0$$

$$\mu_2 = \frac{\Sigma f_i (x_i - 22)^2}{\Sigma f} = \frac{810}{10} = 81$$

$$\mu_3 = \frac{-1440}{10} = -144$$

$$\mu_4 = \frac{148170}{10} = 14817$$

b) i) Mean Time To Failure

$$f(t) = \lambda e^{-\lambda t} \quad t > 0$$

$$E(X) = \int_0^\infty t f(t) dt$$

$$= \int_0^\infty \lambda t e^{-\lambda t}\, dt$$

$$= \lambda \left[\left(\frac{-t e^{-\lambda t}}{\lambda} \right)_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t}\, dt \right]$$

$$= \lambda \left[0 - \left(\frac{e^{-\lambda t}}{\lambda^2} \right)_0^\infty \right]$$

$$= \lambda \left(+ \frac{1}{\lambda^2} \right) = \frac{1}{\lambda}$$

$$E(X^2) = \int_0^\infty t^2 f(t)\, dt$$

$$= \lambda \left[\left(\frac{-t^2 e^{-\lambda t}}{\lambda} \right)_0^\infty + \frac{2}{\lambda} \int_0^\infty t e^{-\lambda t}\, dt \right]$$

$$= \lambda \left[\frac{2}{\lambda} \cdot \frac{1}{\lambda^2} \right]$$

$$= \frac{2}{\lambda^2}$$

ii. $Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

iii. $f(t) = \lambda e^{-\lambda t} \Rightarrow F(t) = \int_0^t f(t)\, dt$

$$= \lambda \int_0^t e^{-\lambda t}\, dt = 1 - e^{-\lambda t}$$

$R(t) = 1 - F(t) \Rightarrow 1 - (1 - e^{-\lambda t})$

$\Rightarrow e^{-\lambda t}$

Hence, failure Rate Function $\lambda(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} \Rightarrow \lambda$

X : Number of defective items

without replacement →

X	0	1	2	3	4
P(X):	$\frac{20C_4}{25C_4}$	$\frac{5C_1 \times 20C_3}{25C_4}$	$\frac{5C_2 \times 20C_2}{25C_4}$	$\frac{5C_3 \times 20C_1}{25C_4}$	$\frac{5C_4}{25C_4}$

$$E(X) = (0)(.383) + (1)(.450) + (2)(.150) + (3)(0.016) + 4(.001)$$

$$E(X) = 0.802$$

with replacement →

Defective items $(p) = \frac{5}{25} = \frac{1}{5}$

Good items $(q) = 1 - \frac{1}{5} = \frac{4}{5}$

X :	0	1	2	3	4
Pi :	$4C_0 q^4$ $= .4096$	$4C_1 pq^3$ $= .4096$	$4C_2 p^2 q^2$ $= 0.1536$	$4C_3 p^3 q$ $= 0.0256$	$4C_4 p^4$ $= 0.0016$

$$E(X) = (0)(.4096) + (1)(.4096) + 2(0.1536) + 3(0.0256) + 4(.0016)$$

$$E(X) = 0 + .4096 + 0.3072 + 0.0768 + 0.0064$$

$$E(X) = 0.8$$

Shilpi
Dr. Shilpi Jain
Submitted

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

POORNIMA
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LECTURE NOTES

Campus: PCE **Course:** B.Tech (CSE) **Class/Section:** 4 sem **Date:** ……………
Name of Faculty: Shilpi Jain **Name of Subject:** SPT **Code:** ……………
Date (Prep.): …………… **Date (Del.):** …………… **Unit No.:** 1 **Lect. No:** ……………

OBJECTIVE: To be written before taking the lecture (Pl. write in bullet points the main topics/concepts etc., which will be taught in this lecture)

Normal Distribution
Variable (problems)

IMPORTANT & RELEVANT QUESTIONS:

prove that Normal Distribution is the limiting case of Binomial Distribution

FEED BACK QUESTIONS (AFTER 20 MINUTES):

Mean $x = 75$, $s.d = 15$
find the workers who receive working wages (i) More than 90 (ii) less than 45

OUTCOME OF THE DELIVERED LECTURE: To be written after taking the lecture (Pl. write in bullet points about students' feedback on this lecture, level of understanding of this lecture by students etc.)

student should be learn about the continuous Distribution.

REFERENCES: Text/Ref. Book with Page No. and relevant Internet Websites:

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Campus: P.C.E. Course: B. Tech Class/Section: CS Date: ....
Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A
Date (Prep.): .... Date (Del.): .... Unit No.: .... Lect. No: 15

OBJECTIVE: To be written before taking the lecture (Pl. write in bullet points the main topics/concepts etc., which will be taught in this lecture)

Normal distribution
Variance (problems)
~~M.g.f~~

IMPORTANT & RELEVANT QUESTIONS:

prove that Normal distribution is the limiting case of Binomial distribution.

FEED BACK QUESTIONS (AFTER 20 MINUTES):

Q. Mean $\mu = 75$. s.d = 15
find the workers who receive weekly wages
wages (i) More than 90 (ii) less than 45?
Q. The 31% items are under 45 & 8% items are over 64
find mean & variance of Normal distribution.

OUTCOME OF THE DELIVERED LECTURE: To be written after taking the lecture (Pl. write in bullet points about students' feedback on this lecture, level of understanding of this lecture by students etc.)

Students learn to solve the problems on continuous distribution.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

REFERENCES: Text/Ref. Book with Page No. and relevant Internet Websites:

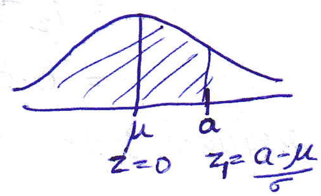
(i) V.K. Kapoor

$45-\mu=\sigma(-0.5)$
$64-\mu=\sigma(1.41)$

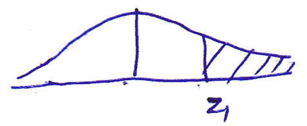
PGC	DETAILED LECTURE NOTES	DATE:..........................
Name of Faculty:	College:	Dept:
Name of Subject with Code:	Branch.:	Class:

formulas:

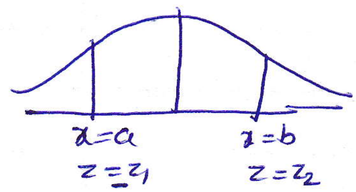
(i) $P(x\le a) = P(-\infty<z<0)+P(0<z<z_1)$
$= \frac{1}{2}+\phi(z_1)$



(ii) $P(x\ge a) = P(0<z<\infty)-P(0<z<z_1)$
$= \frac{1}{2}-\phi(z_1)$

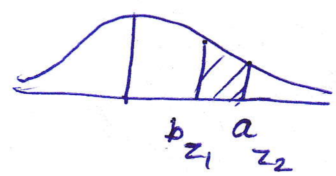


(iii) $P(b\le x\le a) = \phi(z_1)+\phi(z_2)$



(iv) $P(b\le x\le a)$ if $b>\mu,\ a>\mu$

$= P(z_1\le z\le z_2)$
$= \phi(z_2)-\phi(z_1)$



Q. Define Normal distribution. The distribution of weekly wages of 500 workers in a factory is approximately normal with the mean & s.d. of Rs. 75 and Rs. 15 respectively. Find the number of workers who receive weekly wages:
(i) more than 90 (ii) less than 45.

Sol^n: $\mu=75,\ \sigma=15$

$z=\frac{x-\mu}{\sigma}=\frac{x-75}{15}$

for $x=90 \Rightarrow z=\frac{90-75}{15}=1$

for $x=45 \Rightarrow z=\frac{45-75}{15}=-2$

(i) $P(X>90) \Rightarrow P(Z>1) = P(0<z<\infty)-P(0<z<1)$
$= \frac{1}{2}-\phi(1)$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
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COLLEGE OF ENGINEERING

TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 7

Campus: P.C.E. Course: B.Tech Class/Section: II[nd] Year (CS/IT) Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation: .... Scheduled Date of Tut.: .... Actual Date of Tut.: ....

Name of Student: .... Scheduled & Actual Date of H.A. Submission: .... & ....

FIRST 20 MT. CLASS QUESTIONS

Q.1. At a One man barber shop, customers arrive according to the poision distribution with a mean arrival rate of 4 per hour and their hair cutting is exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following:-

W_q (i) Expected time in minutes that a customer has to spend in the queue.

$P(n \geq 5)$ (ii) Probability that there are at least 5 customers in the system.

(iii) Percentage of time the barber ~~a~~ is idle in 8-hour day.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

M/M/1/N, N = 2, $W_s = 21.4$ min

Q.2. In a barber shop which can accommodate 2 people at a time (1 waiting and 1 getting hair cut), customers arrive according to poisson distribution with mean arrival rate 3 per hour. The barber cuts the hair at an average rate of 4 per hour. Find the average number of customers in the shop and the average waiting time in the system and in the queue. $L_s = 0.810$, $P_0 = 0.4324$, $\lambda' = 2.270$

OTHER IMPORTANT QUESTIONS

Q.3. At a petrol pump customer arrive in a Poisson process with an average time of 5 minutes between two arrivals. The time Intervals between services at the pump follow the exponential distribution with mean time of serving one unit being 2 minutes. Find:-

(i) The average queue length and the average number of customers in the system. $L_q = 0.2667$, $L_s = 0.667$

(ii) The time spent by a customer in the queue and in the system.

$W_q = 0.056$ hr $= 3.335$ min

Dr. Mahesh Bundele
B.E./M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Q.4. If for a Period of 2 hours in a day (8-10 AM), trains arrive at the yard, capacity of which is 4 trains, every 20 minutes but the service time remains 36 minutes, then calculate for this period,

(i) The Probability the yard is empty. $(P_0) = 0.04$, $L_s = 3.02$

(ii) The average ~~queue~~ queue ~~length~~ length. (L_q) . $L_q = L_s - \lambda/\mu = 3.02 - 1.8 = 1.22$

(iii) ~~Effective arrival rate.~~ (λ')

Q.5. The arrival of large jobs at a computing center forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distributed with mean 20 minutes. Only four large jobs can be accomodated in the system at a time. Assuming that the fraction of computing power utilized by smaller job is negligible, determine the probability that a large job is turned away because of lack of storage space.

$N=4$.

~~Q.6~~

A large job is turned away if the number of jobs in the system is $N=4$

$P_4 = P_0 \rho^4 = 0.076$.

Ans.1. (i) $W_q = 48\text{ min} = 0.8\text{ hr}$.

(ii) $P(n \geq 5) = \rho^5 = (0.8)^5 = 0.328$

(iii) $P_0 = 0.2 = 20\%$.

$8 \times 0.2 = 1.6\text{ hrs}$.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET SHEET No.............

Campus: .... Course: .... Class/Section: .... Date: ....

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Date of Tut. SheetPreparation:.........................Scheduled Date of Tut.:........................Actual Date of Tut. :.....................

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FIRST 20 MT. CLASS QUESTIONS

Ans.1. $\lambda = 4$ per hour, $\mu = 1/12$ per min

$\lambda = \frac{4}{60} = \frac{1}{15}$ per min.

(i) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = 48$ min. $\lambda/\mu = 0.8$

(ii) $P(n \geq 5) = \rho^5 = (0.8)^5 = 0.328$.

(iii) $P_0 = P$(no customer in shop) $= 1 - \lambda/\mu = 1 - 0.8 = 0.2 = 20\%$.

Hence in 8 hour day $= 8 \times 0.2 = 1.6$ hrs.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Ans.2. $\lambda = 3$ per hour $N = 2$

$\mu = 4$ per hour

$\rho = 3/4 = 0.75$

$P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}} = 0.4324 = \frac{0.25}{0.5781}$

$L_s = \sum_{n=0}^{2} n P_n = 0.810$ customers.

$\lambda' = \lambda(1-P_2) = 3(1-0.2432) = 2.270$

OTHER IMPORTANT QUESTIONS

$W_s = \frac{L_s}{\lambda'} = 0.357$ hours $= 21.4$ min.

$L_q = L_s - \lambda/\mu = 0.2424$ cus.

$W_q = \frac{L_q}{\lambda'} = 0.1067$ hours $= 6.40$ min.

Q.6. $\lambda = 3$ per hour, $\mu = 4$ trucks per hour

$\rho = 0.75$

(i) $P(n \geq 1) = 1 - P_0 = 0.75$

(ii) $W_s = \frac{1}{\mu-\lambda} = 1$ hour.

Ans.3. $\lambda = \frac{1}{5}$ per min

$\lambda = \frac{60}{5} = 12$ per hour

$\mu = 1/2$ per min.

$= \frac{60}{2} = 30$ per hour

(i) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = 0.2667 \approx 1$

(ii) $L_s = \frac{\lambda}{\mu-\lambda} = 0.667$

(iii) $W_s = \frac{L_s}{\lambda} = 0.056$ hr. $= 3.335$ min

$W_q = \frac{0.2667}{12} = 0.022$ hour $= 1.3$ min.

Q.4. $N = 4$

$\lambda = \frac{1}{20}$ per min., $\mu = \frac{1}{36}$ per min

$\rho = 1.8$

(i) $P_0 = 0.04$

$L_s = 3.02 \approx 3$ trains

Q.5. $N = 4$

$\lambda = 1/20$ per min, $\mu = 1/20$ per min.

$\rho = \lambda/\mu = 0.667$

A large % will be turned away if the ... jobs in the sy...

Dr. Mahesh Bundele
B.E./M.E./Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

POORNIMA COLLEGE OF ENGINEERING

Unit 1.

TUTORIAL SHEETS – 3 F

CO-1.

TUTORIAL SHEET SHEET No. 3 F

Campus: PCE Course: B.Tech Class/Section: IInd Year Date: ....

Name of Faculty: Dr. Shilpi jain Name of Subject: SPT Code: 4CS3A

Date of Tut. Sheet Preparation:................ Scheduled Date of Tut.:................ Actual Date of Tut. :................

Name of Student:................ Scheduled & Actual Date of H.A. Submission:................ &................

FIRST 20 MT. CLASS QUESTIONS

Q.1. If μ_r' is the rth moment about origin prove that $\mu_r' = \sum_{j=1}^{r} {}^{r-1}C_{j-1}\, \mu_{r-j}'\, k_j$, where k_j is the jth cumulant.

Q.2. If the joint prob. density of X & Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y), & 0<x<1,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find marginal density of Y.

(ii) Conditional density of X given Y=1.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. If the reliability R(t) of a system is given by $R(t) = 3e^{-2\lambda t} - 2e^{-3\lambda t}$, $t>0$. Find its mean time to failure.

Ans. $5/6\lambda$.

OTHER IMPORTANT QUESTIONS

Q.4. Let the joint pmf of X & Y be given as $P(0,0) = 0.4$, $P(0,1) = 0.2$, $P(1,0) = 0.1$, $P(1,1) = 0.3$. Find conditional prob. mass function of X given Y=1.

Ans. $\frac{2+x}{5}$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 3B

Campus: PCE Course: B.Tech Class/Section: IInd Year Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A

Date of Tut. Sheet Preparation: .... Scheduled Date of Tut.: .... Actual Date of Tut.: ....

Name of Student: .... Scheduled & Actual Date of H.A. Submission: .... & ....

FIRST 20 MT. CLASS QUESTIONS

Q.1. If μ'_r is the r^{th} moment about Origin prove that $\mu'_r = \sum_{j=1}^{r} {}^{r-1}C_{j-1}\, \mu'_{r-j} k_j$, where k_j is the j^{th} cumulant.

Q.2. If the Joint p.d. of X & Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y), & 0<x<1,\ 0<y<2 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find marginal density of Y.

(ii) Conditional density of X given Y=1.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. The failure rate for a certain type of component is $\lambda(t) = at$, $t \geq 0$ where $a>0$ & is constant. Find the component of reliability & hence find its mean ~~to variance~~ time to failure.

Ans. $e^{-at^2/2}$, $\sqrt{\frac{\pi}{2a}}$

OTHER IMPORTANT QUESTIONS

Q.4. Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn & Y denotes the number of red balls drawn, find the joint prob. distribution of (X,Y). Marginal Prob. distribution of X & Y. Conditional prob. distribution of Y given X=1.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 3C

Campus: PCE Course: B.Tech Class/Section: IInd Year Date: ……………

Name of Faculty: Dr. Shilpi jain Name of Subject: SPT Code: 4CS3A

Date of Tut. Sheet Preparation: ……… Scheduled Date of Tut.: ……… Actual Date of Tut.: ………

Name of Student: ……… Scheduled & Actual Date of H.A. Submission: ……… & ………

FIRST 20 MT. CLASS QUESTIONS

Q.1. If μ_r' is the rth ~~cumulant~~ moment about origin prove that $\mu_r' = \sum_{j=1}^{r} {}^{r-1}C_{j-1}\, \mu'_{r-j}\, k_j$, where k_j is the ~~rth~~ jth cumulant.

Q.2 If the joint pdf of X & Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y), & 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

(i) Find marginal density of 'y'.

(ii) Conditional density of X given Y=1

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. If the r.v X has a linear failure rate function $\lambda(t) = a+bt$. Find its distribution and density function.

Ans. $f(t) = (a+bt)\, e^{-at-\frac{bt^2}{2}}$

$F(t) = 1 - e^{-at-\frac{bt^2}{2}}$

OTHER IMPORTANT QUESTIONS

Q.4 The joint Prob. ~~density~~ mass function of two discrete ~~continuous~~ random variates (X, Y) is given by

$$p(x,y) = \begin{cases} \frac{xy}{27}, & \text{for } x=1,2,\ y=2,3,4 \\ 0, & \text{elsewhere} \end{cases}$$

Prove that X & Y are Independent.

Q.4. If the Joint Prob. density function

$f(x,y) = \{ 2 \quad 0<x<1,\ 0<y<1$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 3E

Campus: PCE Course: B.Tech Class/Section: IInd Year Date: ....

Name of Faculty: Dr. Shilpi jain Name of Subject: S.P.T. Code: 4CS3A

Date of Tut. SheetPreparation:................Scheduled Date of Tut.:................Actual Date of Tut.:................

Name of Student:................Scheduled & Actual Date of H.A. Submission:................&................

FIRST 20 MT. CLASS QUESTIONS

Q.1. If μ_r' is the rth moment about Origin, prove that
$\mu_r' = \sum_{j=1}^{r} {}^{r-1}C_{j-1} \, \mu'_{r-j} k_j$, where k_j is the j^{th} cumulant.

Q.2. If the joint prob. density of X & Y is given by
$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y), & 0<x<1, \; 0<y<2 \\ 0, & \text{otherwise} \end{cases}$$
(i) Find marginal density of Y.
(ii) Conditional density of X given Y=1.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3 Let the joint pmf of X & Y be given as
$P(0,0) = 0.4$, $P(0,1) = 0.2$, $P(1,0) = 0.1$, $P(1,1) = 0.3$.
Find conditional Prob. mass function of Y given X=1.

OTHER IMPORTANT QUESTIONS

Q.4 The failure rate for a ~~cer~~ certain type of component is $\lambda(t) = at$, $t \geq 0$ where $a > 0$ & is constant. Find the component of reliability & hence find its mean time to failure.

Ans. $e^{-at^2/2}$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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ज्ञानम् विना न किमपि सौख्यम्

TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 3A

Campus: PCE Course: B.Tech. Class/Section: II^{nd} Year Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: SPI Code: 4CS3A

Date of Tut. Sheet Preparation:.......................... Scheduled Date of Tut.:........................ Actual Date of Tut. :....................

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FIRST 20 MT. CLASS QUESTIONS

Q.1. If μ'_r is the r^{th} moment about origin, prove that $\mu'_r = \sum_{j=1}^{r} {}^{r-1}C_{j-1}\, \mu'_{r-j}\, k_j$ where k_j is the j^{th} cumulant.

Q.2. If the joint p.d. of X & Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{for } 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

(i) Find marginal density of Y
(ii) Conditional density of X given Y=1

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. Suppose a computer center owner want to purchase 3 computers from a sale in which there are 3 new computers, 4 used but still working & 5 defective but can be brought in working condition after minor repair. If X and Y denote new & used but still working computers in the lot of 3 purchased computers. Find (i) The joint prob. mass function of X and Y.
(ii) The marginal prob. of X & Y. (iii) The conditional Prob. of X given Y=1.

OTHER IMPORTANT QUESTIONS

Q.4. If $f(t)$ be the pdf of time to failure T of a system & $h(t)$ be hazard rate function. Find $h(t)$ & MTTF ~~when~~ when $f(t) = \lambda^2 t e^{-\lambda t}$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET

SHEET No. 7 (A)

Campus: P.C.E. Course: B.Tech Class/Section: IInd Year (CS/IT) Date: ....
Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A
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Name of Student:..................................Scheduled & Actual Date of H.A. Submission:..........................&..........................

FIRST 20 MT. CLASS QUESTIONS

Q.1. At a One man barber shop, customers arrive according to the poision distribution with a mean arrival rate of 4 per hour and their hair cutting is exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following:-

(i) Expected time in minutes that a customer has to spend in the queue.

(ii) Probability that there are at least 5 customers in the system.

(iii) Percentage of time the barber ~~a~~ is idle in 6-hour day.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.2. In a barber shop which can accommodate 2 people at a time (1 waiting and 1 getting hair cut), customers arrive according to poison distribution with mean arrival rate 3 per hour. The barber cuts the hair at an average rate of 4 per hour. Find the average number of customers in the shop and the average waiting time in the system and in the queue.

OTHER IMPORTANT QUESTIONS

Q.3. At a petrol pump customer arrive in a Poisson process with an average time of 5 minutes between two arrivals. The time Intervals between services at the pump follow the exponential distribution with mean time of serving one unit being 2 minutes. Find:-

(i) The average queue length and the average number of customers in the system.

(ii) The time spent by a customer in the queue ... system.

Q.4. If for a Period of 2 hours in a day (8-10 AM), trains arrive at the yard, capacity of which is 4 trains, every 20 minutes but the service time remains 36 minutes, then calculate for period,

(i) The Probability the yard is empty.
(ii) The average queue length.
(iii) Effective arrival rate.

Q.5. The arrival of large jobs at a computing center forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distributed with mean 20 minutes. Only four large jobs can be accomodated in the system at a time. Assuming that the fraction of computing power utilized by smaller jobs is negligible, determine the probability that a large job is turned away because of lack of storage space.

Q.6 A milk plant distributes its products by trucks, loaded at the loading dock. It has its own fleet plus the trucks of a transport company are also used. This company has complained that sometimes the trucks have to wait in the queue and thus the company loses money. The company has asked the management either to go in for a second loading dock or discount prices equivalent to waiting time. The data available is as follows: average arrival rate = 3 per hour and average service rate = 4 per hour.
The transport company has provided 40% of the total number of trucks. Determine

(i) The probability that the truck has to wait.
(ii) The waiting time of a truck.

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TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 7 B.

Campus: P.C.E. Course: B.Tech Class/Section: IInd Year (CS/IT) Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: SPT Code: 4CS3A/4IT3A

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FIRST 20 MT. CLASS QUESTIONS

Q.1. At a One man barber shop, customers arrive according to the poision distribution with a mean arrival rate of 4 per hour and their hair cutting is exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following:-

(i) Expected time in minutes that a customer has to spend in the queue.

(ii) Probability that there are at least 5 customers in the system.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

(iii) Percentage of time the barber is idle in 8-hour day.

Q.2. In a barber shop which can accommodate 2 people at a time (1 waiting and 1 getting hair cut), customers arrive according to poisson distribution with mean arrival rate 3 per hour. The barber cuts the hair at an average rate of 4 per hour. Find the average number of customers in the shop and the average waiting time in the system and in the queue.

...HER IMPORTANT QUESTIONS

Q.3. At a petrol pump customer arrive in a Poisson process with an average time of 5 minutes between two arrivals. The time Intervals between services at the pump follow the exponential distribution with mean time of serving one unit being 2 minutes. Find:-

(i) The average queue length and the average number of customers in the system.

(ii) The time spent by a customer in the que[ue] ... system.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Q.4. If for a Period of 2 hours in a day (8-10 AM), trains arrive at the yard, capacity of which is 4 trains, every 20 minutes but the service time remains 36 minutes, then calculate for period,

(i) The Probability the yard is empty.
(ii) The average queue length.
(iii) Effective arrival rate.

Q.5. The arrival of large jobs at a computing center forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distributed with mean 20 minutes. Only four large jobs can be accomodated in the system at a time. Assuming that the fraction of computing power utilized by smaller jobs is negligible, determine the probability that a large job is turned away because of lack of storage space.

Q.6 Car arrive at a petrol pump with exponential inter-arrival times having mean $\frac{1}{2}$ minute. The attendant takes on average of $\frac{1}{5}$ minute per car to supply petrol. The service times being exponentially distributed. Determine

(i) the average number of cars in the system.
(ii) the average number of cars in the waiting line.

Ans. (i) 2 Cars
(ii) $4/3 \approx 1$ car.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET

SHEET No. 7 C

Campus: P.C.E. Course: B.Tech Class/Section: IInd Year (CS/IT) Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A

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FIRST 20 MT. CLASS QUESTIONS

Q.1. At a One man barber shop, customers arrive according to the poision distribution with a mean arrival rate of 4 per hour and their hair cutting is exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following:-

(i) Expected time in minutes that a customer has to spend in the queue.

(ii) Probability that there are at least 5 customers in the system.

(iii) Percentage of time the barber is idle in 8-hour day.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.2. In a barber shop which can accommodate 2 people at a time (1 waiting and 1 getting hair cut), customers arrive according to poisson distribution with mean arrival rate 3 per hour. The barber cuts the hair at an average rate of 4 per hour. Find the average number of customers in the shop and the average waiting time in the system and in the queue.

OTHER IMPORTANT QUESTIONS

Q.3. At a petrol pump customer arrive in a Poisson process with an average time of 5 minutes between two arrivals. The time Intervals between services at the pump follow the exponential distribution with mean time of serving one unit being 2 minutes.

Find:-

(i) The average queue length and the average number in the system.

(ii) The time spent by a customer in the queue [illegible] system.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Q.4. If for a Period of 2 hours in a day (8-10 AM), trains arrive at the yard, capacity of which is 4 trains, every 20 minutes but the service time remains 36 minutes, then calculate for period,

(i) The Probability the yard is empty.

(ii) The average queue length.

(iii) Effective arrival rate.

Q.5. The arrival of large jobs at a computing center forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distributed with mean 20 minutes. Only four large jobs can be accomodated in the system at a time. Assuming that the fraction of computing power utilized by smaller job is negligible, determine the probability that a large job is turned away because of lack of storage space.

Q.6 Customers arrive at a sales counter manned by a single person according to a poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distibution with a mean of 100 seconds. Find the average waiting time of a customer.

$W_q = 125$ seconds

$W_s = 225$ seconds.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
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TUTORIAL SHEETS

TUTORIAL SHEET SHEET No....7...

Campus: ..P.C.E..... Course: ..B.Tech..... Class/Section: ..IInd Year (CS/IT) Date: ....

Name of Faculty: Dr. Shilpi Jain.... Name of Subject: ..S.P.T.............. Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation:.........................Scheduled Date of Tut.:.......................Actual Date of Tut. :...................

Name of Student:...................................Scheduled & Actual Date of H.A. Submission:...........................&..........................

FIRST 20 MT. CLASS QUESTIONS	Q.1. At a One man barber shop, customers arrive according to the poision distribution with a mean arrival rate of 4 per hour and their hair cutting is exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following:- (i) Expected time in minutes that a customer has to spend in the queue.
2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS	(ii) Probability that there are at least 5 customers in the system. (iii) Percentage of time the barber is idle in 8-hour day. Q.2. In a barber shop which can accommodate 2 people at a time (1 waiting and 1 getting hair cut), customers arrive according to poisson distribution with mean arrival rate 3 per hour. The barber cuts the hair at an average rate of 4 per hour. Find the average number of customers in the shop and the average waiting time in the system and in the queue.
OTHER IMPORTANT QUESTIONS	Q.3. At a petrol pump customer arrive in a Poisson process with an average time of 5 minutes between two arrivals. The time Intervals between services at the pump follow the exponential distribution with mean time of serving one unit being 2 minutes. Find:- (i) The average queue length and the average number of customers in the system. (ii) The time spent by a customer in the queue

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
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Poornima College of Engineering
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Sitapura, JAIPUR

Q.4. If for a Period of 2 hours in a day (8-10 AM), trains arrive at the yard, capacity of which is 4 trains, every 20 minutes but the service time remains 36 minutes, then calculate for period,

(i) The Probability the yard is empty.

(ii) The average queue length.

(iii) Effective arrival rate.

Q.5. The arrival of large jobs at a computing center forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distibuted with mean 20 minutes. Only four large jobs can be accomodated in the system at a time. Assuming that the fration of computing power utilized by smaller job is negligible, determine the probability that a large job is turned away because of lack of storage space.

Q.6 Barber A takes 15 minutes to complete one hair cut. Customers arrive in his shop at an average rate of one every 30 minutes and the arrival process is Poisson. Barber B takes 25 minutes to complete one hair cut and customers arrive in his shop at an average rate of one every 50 minutes, the arrival process being Poisson during steady state.

(i) Where would you expect the bigger queue?

(ii) Where would you require more waiting times, including the time required to complete a hair cut?

Ans (i) Barber A (ii) Barber B

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

POORNIMA
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TUTORIAL SHEETS

TUTORIAL SHEET

SHEET No. 7 E

Campus: P.C.E. Course: B.Tech Class/Section: II nd Year (CS/IT) Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation:........................Scheduled Date of Tut.:........................Actual Date of Tut.:....................

Name of Student:....................................Scheduled & Actual Date of H.A. Submission:..........................&..........................

FIRST 20 MT. CLASS QUESTIONS

Q.1. At a One man barber shop, customers arrive according to the poision distribution with a mean arrival rate of 4 per hour and their hair cutting is exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following:-

(I) Expected time in minutes that a customer has to spend in the queue.

(II) Probability that there are at least 5 customers in the system.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

(III) Percentage of time the barber ~~d~~ is idle in 8-hour day.

Q.2. In a barber shop which can accommodate 2 people at a time (1 waiting and 1 getting hair cut), customers arrive according to poisson distribution with mean arrival rate 3 per hour. The barber cuts the hair at an average rate of 4 per hour. Find the average number of customers in the shop and the average waiting time in the system and in the queue.

OTHER IMPORTANT QUESTIONS

Q.3. At a petrol pump customer arrive in a Poisson process with an average time of 5 minutes between two arrivals. The time Intervals between services at the pump follow the exponential distribution with mean time of serving one unit being 2 minutes. Find:-

(i) The average queue length and the average number of customers in the system.

(ii) The time spent by a customer in the queue ... system.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Q.4. If for a Period of 2 hours in a day (8-10 AM), trains arrive at the yard, capacity of which is 4 trains, every 20 ~~mins~~ minutes but the service time remains 36 minutes, then calculate for period,

(i) The Probability the yard is empty.

(ii) The average queue length.

(iii) Effective arrival rate.

Q.5. The arrival of large jobs at a computing center forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distributed with mean 20 minutes. Only four large jobs can be accomodated in the system at a time. Assuming that the fraction of computing power utilized by smaller job is negligible, determine the probability that a large job is turned away because of lack of storage space.

Q.6 Customers arrive at a box office with one ticket window according to a Poisson input process with mean rate of 30 per hour. The time required to serve a customer is exponential with mean of 90 seconds. Calculate

(i) What is the average line length?

(ii) What " " queue length?

(iii) What is the average waiting time in the queue and in the system.

Ans. (i) 3 cus. (ii) 2.25 cus. (iii) 4 min (iv) 6 mins.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 7 F

Campus: PCE Course: B.Tech Class/Section: IInd Year (CS/IT) Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: SPT Code: 4CS3A/4IT3A

Date of Tut. SheetPreparation:........................Scheduled Date of Tut.:........................Actual Date of Tut. :....................

Name of Student:......................................Scheduled & Actual Date of H.A. Submission:...........................&..........................

FIRST 20 MT. CLASS QUESTIONS

Q.1. At a One man barber shop, customers arrive according to the poision distribution with a mean arrival rate of 4 per hour and their hair cutting is exponentially distributed with an average hair cut taking 12 minutes. There is no restriction in queue length. Calculate the following:-

(i) Expected time in minutes that a customer has to spend in the queue.

(ii) Probability that there are at least 5 customers in the system.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

(iii) Percentage of time the barber is idle in 8-hour day.

Q.2. In a barber shop which can accommodate 2 people at a time (1 waiting and 1 getting hair cut), customers arrive according to poisson distribution with mean arrival rate 3 per hour. The barber cuts the hair at an average rate of 4 per hour. Find the average number of customers in the shop and the average waiting time in the system and in the queue.

IMPORTANT QUESTIONS

Q.3. At a petrol pump customer arrive in a Poisson process with an average time of 5 minutes between two arrivals. The time Intervals between services at the pump follow the exponential distribution with mean time of serving one unit being 2 minutes. Find:-

(i) The average queue length and the average Dr. Mahesh Bundele B.E., M.E., Ph.D. Director Poornima College of Engineering ISI-6, RIICO Institutional Area Sitapura, JAIPUR in the system.

Q.4. If for a Period of 2 hours in a day (8-10 AM), trains arrive at the yard, capacity of which is 4 trains, every 20 minutes but the service time remains 36 minutes, then calculate for period,

(i) The Probability the yard is empty.

(ii) The average queue length.

(iii) Effective arrival rate.

Q.5. The arrival of large jobs at a computing center forms a Poisson process with rate two per hour. The service times of such jobs are exponentially distributed with mean 20 minutes. Only four large jobs can be accomodated in the system at a time. Assuming that the fraction of computing power utilized by smaller job is negligible, determine the probability that a large job is turned away because of lack of storage space.

Q.6 A maintenance service facility has Poisson arrival rate = 3 per day and negative exponential service time with service rate 6 per day. It operates on a first come first served queue discipline. Find

(i) Mean number of the customer in the system.

(ii) Mean waiting time in the queue.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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LECTURE NOTES

Campus: PCE Course: B.Tech(CSE) Class/Section: 4th Date: ....
Name of Faculty: Shilpi Jain Name of Subject: SPT Code: ....
Date (Prep.): .... Date (Del.): .... Unit No.: 2 Lect. No: ....

OBJECTIVE: To be written before taking the lecture (Pl. write in bullet points the main topics/concepts etc., which will be taught in this lecture)

Probability

IMPORTANT & RELEVANT QUESTIONS:

Numerical Question of probability

FEED BACK QUESTIONS (AFTER 20 MINUTES):

A man is known to speak truth 3 out of 4 times. find the probability that it is actually a six.

OUTCOME OF THE DELIVERED LECTURE: To be written after taking the lecture (Pl. write in bullet points about students' feedback on this lecture, level of understanding of this lecture by students etc.)

Student should be learn about the probability.

REFERENCES: Text/Ref. Book with Page No. and relevant Internet Websites:

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEETS

TUTORIAL SHEET

SHEET No. 2

Campus: P.C.E. Course: B. Tech Class/Section: IInd Year Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation:........................Scheduled Date of Tut.:........................Actual Date of Tut.:........................

Name of Student:..Scheduled & Actual Date of H.A. Submission:............................&............................

FIRST 20 MT. CLASS QUESTIONS

Q.1. Three balls are drawn at random without replacement from a box containing 2 white, 3 Red & 4 Black balls. If X denotes the number of white balls drawn & Y denotes the number of red balls drawn find the Joint Probability distribution of (X,Y). and Marginal prob. distribution for X & Y. conditional distribution of X given Y=1.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.2. A card is drawn from a pack of 52 cards, If the value of a card be according to its denomination respectively 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, where each face card of king, queen and Jack be assigned the number 10 each. find the expected value of the number of points on the cards.

Ans. 6.5

OTHER IMPORTANT QUESTIONS

Q.3. If $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(i) Show that f(x) is Pdf.

(ii) Find its distribution function.

Ans. $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^2/2}, & x \geq 0 \end{cases}$

Q.4. A r.v X has the following prob. distribution:-

X :	0	1	2	3	4	5	6	7	8
P(X) :	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Find a

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Unit- I.

TUTORIAL SHEETS

CO1

TUTORIAL SHEET

SHEET No. 1 A

Campus: P.C.E. Course: B.Tech. Class/Section: CS/IT Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation:.......................... Scheduled Date of Tut.:.......................... Actual Date of Tut. :......................

Name of Student:.. Scheduled & Actual Date of H.A. Submission:............................&..............................

FIRST 20 MT. CLASS QUESTIONS

Q.1 An urn contain 3 Red + 2 black balls, Two balls are drawn. Find the Prob. that both are of the same colour.

Ans. 2/5.

Q.2. Box A_1 contains 3 White + 3 Black balls & A_2 contains 4 white + 3 Black. One ball is transferred from A_1 to A_2 & After it a ball drawn from A_2 find
(i) The chance that it is white.
(ii) Find the conditional prob. that the transferred ball was also white.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. The chances that a doctor will diagnose a disease correctly is 70%. The chances of death of a patient after correct diagnosis is 35% while after wrong diagnosis it is 80%. If a patient dies after ~~wrong diagnosis~~ taking treatment, find the prob. that he was diagnosis
(i) Wrongly (ii) Correctly

Ans. (i) $\frac{48}{97}$ (ii) $\frac{49}{97}$

OTHER IMPORTANT QUESTIONS

Q.4. A lot of IC chips contains 2% defective chips. The chips are tested before delivery. The tester's report is not 100% reliable. $P\left(\frac{\text{Tester report good}}{\text{chip actually good}}\right) = 0.95$.

& $P\left(\frac{\text{Tester report defective}}{\text{chip is actually defective}}\right) = 0.94$

If the tester reports 'defective' to a chip find the prob. that the chip is actually defective.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Unit-1
CO1

TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 1 F

Campus: P.C.E. Course B. Tech Class/Section: CS/IT Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation:......................... Scheduled Date of Tut.:......................... Actual Date of Tut.:.....................

Name of Student:...................................... Scheduled & Actual Date of H.A. Submission:.......................... &..........................

FIRST 20 MT. CLASS QUESTIONS	Q.1. An Urn contain 3 Red + 2 black balls. Two balls are drawn. Find the Prob. that both are of the same colour. Ans. 2/5. Q.2. Box A_1 contains 3 white + 3 Black Balls & A_2 contains 4 white + 3 Black. One ball is transferred from A_1 to A_2 & after it a ball drawn from A_2 find (i) The chance that it is white. 9/16 (ii) Find the conditional prob. that the transferred ball was also white 5/9
2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS	Q.3 Out of the two types of tubes A & B installed in a television set, three quarters of the tubes are of the type A. The prob. that the tube of type A will fail with in one year is $1/8$ & that of type B will fail is $1/4$. If a set fails within a year, what is the Prob. that tube of type B was installed in it. Ans. 2/5.
OTHER IMPORTANT QUESTIONS	Q.4. A man is known to speak ~~3 out~~ truth 3 out of 4 times. He throws a dice and reports it is 6. Find the probability that it is actually a six. Ans. 3/8.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Unit - I
CO-1

TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 1 D

Campus: PCE Course: B.Tech Class/Section: CS/I.T. Date: ....
Name of Faculty: Dr. Shilpi Jain. Name of Subject: S.P.T. Code: 4CS3A/4IT3A

Date of Tut. SheetPreparation:.......................Scheduled Date of Tut.:.....................Actual Date of Tut. :....................

Name of Student:.....................................Scheduled & Actual Date of H.A. Submission:..........................&...........................

FIRST 20 MT. CLASS QUESTIONS	Q.1. An urn contain 3 Red + 2 black balls. Two balls are drawn. Find the Prob. that both are of the same colour. Ans. $2/5$. Q.2. Box A_1 contains 3 white + 3 Black balls & A_2 contains 4 White + 3 Black. One ball is transferred from A_1 to A_2 & After it a ball drawn from A_2 find (i) The chance that it is white. (ii) Find the conditional prob. that the transferred ball was also white. Ans. $9/16$, $5/9$
2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS	Q.3. Three factories A, B, C does 30%, 50% & 20% production of certain item. Out of their production 8%, 5% & 10% of the items produced are defective respectively. An Item is purchased & is found to be defective. Find the prob. that it was a product of factory A. Ans. $\frac{24}{69}$.
OTHER IMPORTANT QUESTIONS	Q.4. State and Prove the Baye's Th.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Unit-I
CO1

TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 1 B.

Campus: PCE Course: B.Tech. Class/Section: CS/IT Date: ....
Name of Faculty: Dr. Shilpi Jain Name of Subject: S.P.T. Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation: .... Scheduled Date of Tut.: .... Actual Date of Tut.: ....

Name of Student: .... Scheduled & Actual Date of H.A. Submission: .... & ....

FIRST 20 MT. CLASS QUESTIONS

Q.1. An urn contain 3 Red + 2 black balls. Two balls are drawn. Find the Prob. that both are of the same colour. Ans 2/5.

Q.2. Box A_1 contains 3 white + 3 Black balls & A_2 contains 4 white + 3 Black. One ball is transferred from A_1 to A_2 & after it a Ball drawn from A_2 find
(i) The chance that it is white. Ans 9/16
(ii) Find the conditional prob. that the transferred ball was also white. Ans 5/9

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. In a College there were three Candidates for the position of principal. Mr. X, Y, & Z. whose chances of getting the appointment are in the proportion 4:2:3 respectively. The prob. that Mr. X if selected would introduce new branch of today's requirement in the college is 0.3. The prob. of Mr. Y & Z doing the same respectively 0.5 & 0.8. What is the prob. that there will be new branch in the college next year. Ans. 23/45.

OTHER IMPORTANT QUESTIONS

Q.4. An item is produced by 3 factories A, B, C producing 500, 300 & 200 items respectively. The defective items produced by them are 1%, 2%, 4% respectively. An Item purchased from the market is found to be defective. find the Prob. for it is to be purchased by A.

Q.5. State & Prove the Baye's Th.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Unit-I
CO1

TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 1 C

Campus: P.C.E. Course: B.Tech Class/Section: CS/IT Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: SPT Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation:.......................... Scheduled Date of Tut.:.......................... Actual Date of Tut.:......................

Name of Student:.. Scheduled & Actual Date of H.A. Submission:.............................&.............................

FIRST 20 MT. CLASS QUESTIONS

Q.1. An urn contain 3 Red + 2 black balls. Two balls are drawn. Find the Prob. that both are of the same colour. Ans. 2/5.

Q.2. Box A_1 contains 3 White + 3 Black balls & A_2 contains 4 white + 3 Black. One ball is transferred from A_1 to A_2 & after it a Ball drawn from A_2. Find
(i) The chance that it is white.
(ii) Find the conditional prob. that the transferred ball was also white.
Ans. (i) 9/16 (ii) 5/9.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. A company selected Engineer's through campus Interview from an institution. Out of the total selection made from electronics, Computer & Mechanical 60%, 30% & 10% are from respectively. If 9%, 20% & 6% of selected students do not join the company. What is the prob. that who do not join is from Computer stream.

Ans. 0.5

OTHER IMPORTANT QUESTIONS

Q.4. States & Prove the Baye's Th.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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Unit - I
CO1

TUTORIAL SHEETS

TUTORIAL SHEET SHEET No. 1E

Campus: PCE Course: B. Tech. Class/Section: II nd year Date: ....

Name of Faculty: Dr. Shilpi Jain Name of Subject: SPT Code: 4CS3A/4IT3A

Date of Tut. Sheet Preparation: .... Scheduled Date of Tut.: .... Actual Date of Tut.: ....

Name of Student: .... Scheduled & Actual Date of H.A. Submission: .... & ....

FIRST 20 MT. CLASS QUESTIONS

Q.1. An urn contain 3 Red + 2 black balls. Two balls are drawn. Find the Prob. that both are of the same colour. Ans. 2/5

Q.2. Box A_1 contains 3 White + 3 Black balls & A_2 contains 4 white + 3 Black. One ball is transferred from A_1 to A_2 & after it a ball drawn from A_2 find

(i) The chance that it is white.

(ii) Find the conditional Prob. that the transferred ball was also white. Ans. 9/16, 5/9

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q.3. A letter is known to have come either LONDON or CLIFTON. On the envelope just two consecutive letters ON are visible. What is the probability that the letter has come from (i) LONDON

(ii) CLIFTON

Ans. 14/19, 5/19

OTHER IMPORTANT QUESTIONS

Q.4. The probabilities that a man makes a certain journey by car, a motor bike or on foot are 1/2, 1/6 and 1/3 respectively. The probabilities of an accident when he uses these means of transport are 1/5, 3/5 and 1/10 respectively. Find the chance of an accident.

Ans. 7/30.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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LECTURE NOTES

Campus: PCE **Course:** B.Tech (CSE) **Class/Section:** 4th **Date:** ....
Name of Faculty: Shilpi Jain **Name of Subject:** SPT **Code:** ....
Date (Prep.): .... **Date (Del.):** .... **Unit No.:** 3 **Lect. No:** ....

OBJECTIVE: To be written before taking the lecture (Pl. write in bullet points the main topics/concepts etc., which will be taught in this lecture)

Bivariate distribution

coefficient of Correlation

IMPORTANT & RELEVANT QUESTIONS:

Numerical Question

FEED BACK QUESTIONS (AFTER 20 MINUTES):

What is correlation coefficient

OUTCOME OF THE DELIVERED LECTURE: To be written after taking the lecture (Pl. write in bullet points about students' feedback on this lecture, level of understanding of this lecture by students etc.)

Student should be learn about the Bivariate distribution & correlation coefficient.

REFERENCES: Text/Ref. Book with Page No. and relevant Internet Websites:

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEET

TUTORIAL SHEET Unit -3 **SHEET No.** 3(A)

Campus: PCE **Course:** B.Tech **Class/Section:** CS (IInd year) **Date:** ....

Name of Faculty: Dr. Shilpi jain **Name of Subject:** SPT **Code:** ....

Date of Tut. Sheet Preparation:........................ Scheduled Date of Tut.:........................Actual Date of Tut. :....................

Name of Student:....................................Scheduled & Actual Date of H.A. Submission:..........................&..........................

FIRST 20 MT. CLASS QUESTIONS

Q1) Calculate the coefficient of correlation between x & y using the following data:

x	-10	-5	0	5	10
y	5	9	7	11	13

Q2) Find the angle between the two lines of regression. Interpret the case when $r = \pm 1, 0$.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q3) For a bivariate distribution
$n = 18$, $\Sigma x^2 = 60$, $\Sigma y^2 = 96$, $\Sigma x = 12$, $\Sigma y = 18$,
$\Sigma xy = 48$
Find the equations of the lines of regression and r

Q4) Calculate the coefficient of correlation between x and y from the following data

x	1	2	3	4	5	6
y	5	8.1	10.6	13.1	16.2	20.0

OTHER IMPORTANT QUESTIONS

Q5) The marks secured by the student in the Mathematics & Statistics are given below.

Roll No	1	2	3	4	5	6	7	8	9
Marks in Maths	10	19	12	17	13	16	25	14	22
Statistics ~~Statistics~~	30	42	45	46	33	34	40	35	

Calculate the rank correlation coefficient

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TUTORIAL SHEET

TUTORIAL SHEET Unit - 3 | SHEET No. 3(A)

Campus: P.C.E. Course: B-Tech | Class/Section: C.S (II^nd year) | Date: 1/2/18

Name of Faculty: Dr. Shilpi Jain | Name of Subject: SPT | Code: 4CS3A

Date of Tut. Sheet Preparation:........ Scheduled Date of Tut.:........ Actual Date of Tut.:........

Name of Student:........ Scheduled & Actual Date of H.A. Submission:........ &........

FIRST 20 MT. CLASS QUESTIONS

Q1) Calculate the coefficient of correlation between x & y using the following data:

x	−10	−5	0	5	10
y	5	9	7	11	13

Q2) Find the angle between the two line of regression interpret the case when $r = \pm 1, 0$

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q3) For a bivariate distribution

$n = 18$; $\Sigma x^2 = 60$; $\Sigma y^2 = 96$; $\Sigma x = 12$; $\Sigma y = 18$

$\Sigma xy = 48$

find the equations of the line of regression and r.

Q4) Fit a straight line to the following set of observations

x	1	2	3	4	5
y	2	4	6	8	10

OTHER IMPORTANT QUESTIONS

Q5) The marks secured by the student in Mathematics & Statistics are given below:-

Roll No	1	2	3	4	5	6	7	8	9
(marks in)									
Maths	10	15	12	17	13	16	25	14	22
Statistics	30	42	45	46	33	34	45	[illegible]	29

Calculate the rank coefficient

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEET

TUTORIAL SHEET Unit-3 | SHEET No. 1(A)

Campus: PCE | Course: B-Tech | Class/Section: C.S (II nd year) | Date: 1/2/18

Name of Faculty: Dr. Shilpi Jain | Name of Subject: S.P.T | Code: 4CS3A

Date of Tut. Sheet Preparation:.......................... Scheduled Date of Tut.:........................Actual Date of Tut. :....................

Name of Student:..Scheduled & Actual Date of H.A. Submission:...........................&............................

FIRST 20 MT. CLASS QUESTIONS

Q1) Calculate the coefficient of correlation between x & y using the following data.

x	−10	−5	0	5	10
y	5	9	7	11	13

Q2) Find the angle between the two lines of regression. Interpret the case when $r = \pm 1, 0$.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q3) If $r = 0$, show the two line of regression are parallel to the axes.

Q4) Calculate the coefficient of correlation between x & y from the following data

x	1	2	3	4	5	6
y	5	8.1	10.6	13.1	16.2	20.0

OTHER IMPORTANT QUESTIONS

Q5) The marks secured by the student in Mathematics & Statistics are given below:

Roll No	1	2	3	4	5	6	7	8	9
(marks in) Maths	10	15	12	17	13	16	25	14	22
Statistics	30	42	45	46	33	34	40	[illegible]	[illegible]

Calculate the rank correlation coefficient

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEET

TUTORIAL SHEET Unit -3 — **SHEET No.** 4(A)

Campus: PCE Course: B.Tech. Class/Section: C.S (II nd year) Date: 1/2/18

Name of Faculty: Dr. Shilpi jain Name of Subject: SPT Code: 4CS3A

Date of Tut. Sheet Preparation:.................. Scheduled Date of Tut.:.................. Actual Date of Tut.:..................

Name of Student:.................. Scheduled & Actual Date of H.A. Submission:..................&..................

FIRST 20 MT. CLASS QUESTIONS

Q1) Calculate the Cofficient of Correlation between x & y using the following data

x	-10	-5	0	5	10
y	5	9	7	11	13

Q2) Find the angle between the two line of regression except the case when $r = \pm 1, 0$

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q3) The marks secured by the students in Mathematic & Statistic are given below

Roll No.	1	2	3	4	5	6	7	8	9
Marks in maths	10	15	12	17	13	16	25	14	22
	4	5	8	3	7	4	1	6	2
marks in Statistic	30	42	45	46	33	34	40	35	39

Calculate the rank Correlation Cofficient

OTHER IMPORTANT QUESTIONS

Q4) Calculate the Cofficient of Correlation between x and y from the following data

x	1	2	3	4	5	6
y	5	8.1	10.6	13.1	16.2	20.0

Q5) If $r = 0$, show that two line of regression are parallel to the axes

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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TUTORIAL SHEET

TUTORIAL SHEET Unit-3 SHEET No. 2(A)

Campus: PCE Course: B.Tech Class/Section: CS (II^{nd} year) Date: 1/2/18

Name of Faculty: Dr. Shilpi jain Name of Subject: SPT Code: ....

Date of Tut. Sheet Preparation:......................... Scheduled Date of Tut.:......................Actual Date of Tut. :...................

Name of Student:....................................Scheduled & Actual Date of H.A. Submission:.........................&.........................

FIRST 20 MT. CLASS QUESTIONS

Q1) Calculate the coefficient of correlation between x & y using the following data:

x	-10	-5	0	5	10
y	5	9	7	11	13

Q2) Find the angle between the two line of regression interpret the case when $r = \pm 1, 0$.

2 HRS. SOLVABLE HOME ASSIGNMENT (H.A.) QUESTIONS

Q3) The marks secured by the student in Mathematics & Statistics are given below.

Roll No	1	2	3	4	5	6	7	8	9
marks in Maths	10	15	12	17	13	16	25	14	22
~~Statistic~~ Statistics	30	42	45	46	33	34	40	35	39

Calculate the rank correlation coefficient.

OTHER IMPORTANT QUESTIONS

Q4) If $r = 0$, show that two line of regression are parallel to the axis

Q5 For a bivariate distribution

$n = 18$; $\Sigma x^2 = 60$; $\Sigma y^2 = 96$, $\Sigma x = 12$ [illegible]

$\Sigma xy = 48$

Find the equations of the line of regression

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COLLEGE OF ENGINEERING

LECTURE NOTES

Campus: PCE **Course:** B.Tech (CS) **Class/Section:** 4th **Date:** ……………
Name of Faculty: Shilpi Jain **Name of Subject:** SPT **Code:** ……………
Date (Prep.): …………………… **Date (Del.):** …………………… **Unit No.:** 4 **Lect. No:** ……………

OBJECTIVE: To be written before taking the lecture (Pl. write in bullet points the main topics/concepts etc., which will be taught in this lecture)

Queueing Theory.
Service Mechanism
Classification of Queuing model

IMPORTANT & RELEVANT QUESTIONS:

Q. Diff. Elements of Queueing Theory.

FEED BACK QUESTIONS (AFTER 20 MINUTES):

Explain the characteristic's of Model.

OUTCOME OF THE DELIVERED LECTURE: To be written after taking the lecture (Pl. write in bullet points about students' feedback on this lecture, level of understanding of this lecture by students etc.)

student should be learn about the Queuing Theory

REFERENCES: Text/Ref. Book with Page No. and relevant Internet Websites:

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

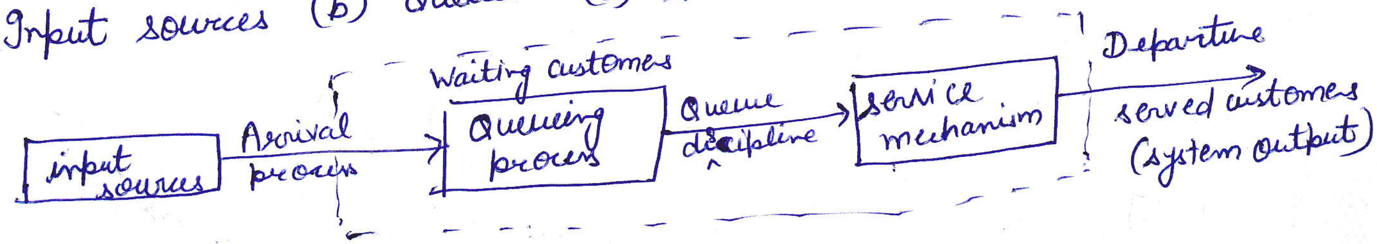
Queueing Theory: Queueing theory is a Collection of mathematical model of various queueing ~~sty~~ systems.

Why queues form: Queues or Waiting lines arise when the demand for a service facility exceeds the capacity of that facility, that is the customers do not get service immediately.

Elements of Queueing System:

The basic queueing process consists of

(a) Input sources (b) Queues (c) service mechanism.



The various elements of the queueing system are

(i) Input sources — its size → The size of the input sources is generally infinite
— arrival pattern → The arrival rate of customer is either constant or random. Therefore, the number of arrival λ per unit time is estimated by poisson dis.

(ii) Waiting line

(iii) Queue discipline → FIFO | LIFO | SIRO
BUS

(iv) Service Mechanism → service time is not constant, for all customers → service time distribution is negative exponential order.

(v) system Output

(vi) Customer behaviour — Balking | Reneging | jockeying

~~Service Mechanism:~~

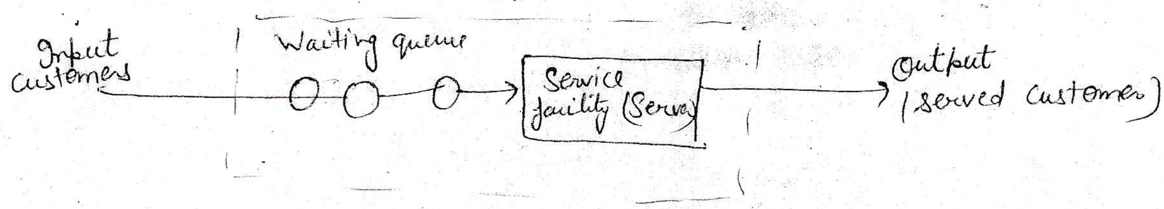
~~1. Single~~

Service Mechanism:

(single queue, single server, single phase system)

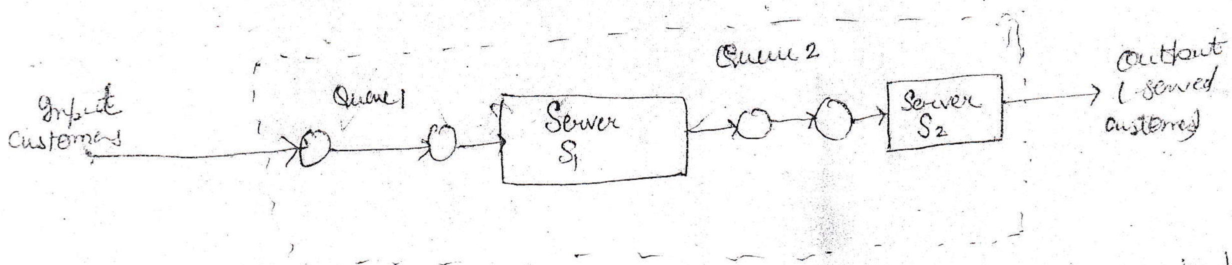
1. Single Channel Single Phase (Single Queue-Single Server):

same service



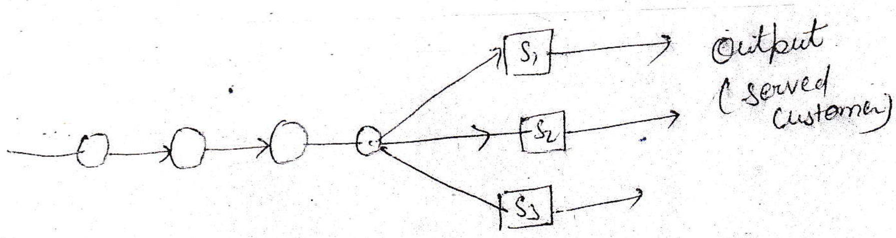
2. Single Channel, Multiphase (Single Queue, Multiple Servers in Series):

~~Multiple~~ phase system

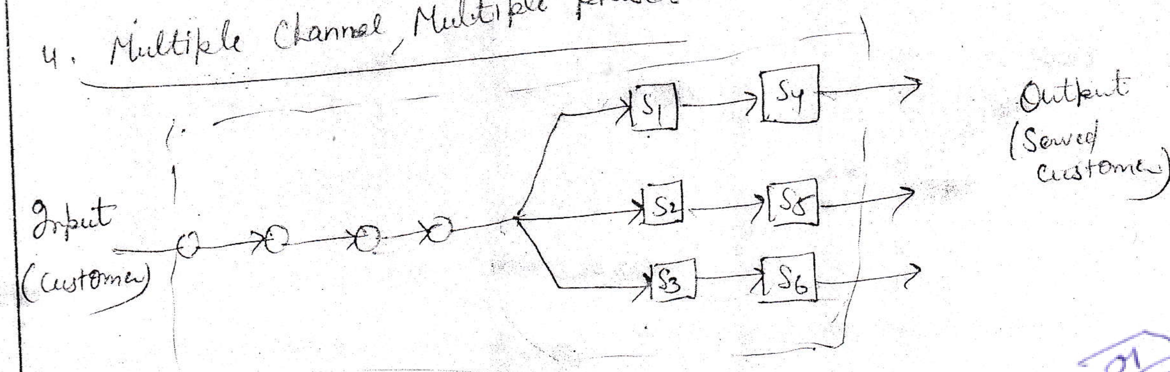


3. Multi channel, Single phase System: (~~single queue~~ Multiple Servers in parallel)

single phase system



4. Multiple Channel, Multiple phase:



PGC	DETAILED LECTURE NOTES	DATE:.........................
Name of Faculty:	College:	Dept:
Name of Subject with Code:	Branch.:	Class:

Pure Birth Process: We could think of a new arrival as a birth & a departure after service as death. So a queueing system can be described with the help of special stochastic process known as birth-death process. Pure Birth Process is one in which only arrivals are counted & no departures take place. Ex. Creation of ~~new~~ Birth certificate for new Born babies. The model in which there may be either zero or one arrival in the interval t to $t+\Delta t$, without any departure during this time interval is called pure birth model.

Let $P_n(t)$ be the Probability of n arrivals in time t. Δt is small time interval which only one customer is arrive.
& λ be average arrival rate.

We make following assumptions:

(i) The Prob. of an arrival in $(t, t+\Delta t)$ is $\lambda \Delta t$.

(ii) The Prob. " no " " $1-\lambda\Delta t$

(iii) The " of more than one arrival in $(t, t+\Delta t)$, i.e time Δt is 0.

(iv) The number of arrivals in non-overlaping intervals are statistically independent. i.e the Process has independent arrivals.

$$P_n(t+\Delta t) = P_{n-1}(t)\lambda\Delta t + P_n(t)(1-\lambda\Delta t), \text{ for } n \geq 1$$

$$P_n(t+\Delta t) - P_n(t) = P_{n-1}(t)\lambda\Delta t - P_n(t)\lambda\Delta t, \text{ for } n \geq 1$$

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \{P_{n-1}(t) - P_n(t)\}\lambda, \quad n \geq 1$$

$$\frac{dP_n(t)}{dt} = \{P_{n-1}(t) - P_n(t)\}\lambda, \quad n \geq 1$$

for $n=0$, $P_{0-1}(t) = 0$ so, we have,

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

PGC **DETAILED LECTURE NOTES** DATE:..........................

Name of Faculty: College: Dept:

Name of Subject with Code: Branch.: Class:

$$\Rightarrow \frac{dP_0(t)}{P_0(t)} = -\lambda\, dt$$

$$\Rightarrow \log P_0(t) = -\lambda t + \log C_0 \Rightarrow P_0(t) = C_0 e^{-\lambda t} \longrightarrow (2).$$

at $t=0$, $P_0(t)=1$

$$\Rightarrow 1 = C_0$$

$$\Rightarrow \boxed{P_0(t) = e^{-\lambda t}} \longrightarrow (3)$$

eqⁿ (1), for $n=1$

$$\frac{dP_1(t)}{dt} = \{P_0(t) - P_1(t)\}\lambda$$

$$\Rightarrow \frac{dP_1}{dt} + P_1(t)\lambda = \lambda e^{-\lambda t} \longrightarrow (4).$$

L.D. eqⁿ of first order.

I.F. $e^{\lambda t}$

solⁿ.

$$P_1(t)e^{\lambda t} = \int e^{\lambda t}\lambda e^{-\lambda t}\,dt + C_1$$

$$P_1(t) = \lambda t e^{-\lambda t} + C_1 e^{-\lambda t}$$

at $t=0$, $P_1(t)=0$

$$0 = 0 + C_1 \Rightarrow C_1 = 0$$

$$\Rightarrow P_1(t) = \frac{(\lambda t)^1}{1!} e^{-\lambda t} \longrightarrow (5)$$

similarly

$$\boxed{P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}}, \quad n = 0, 1, 2, \ldots$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Pure birth Process: We could think of a new arrival as a birth & a departure after service as death. So a queueing system can be described ~~after service is death~~. with the help of special stochastic process known as birth - death process. Pure birth Process is one in which only arrivals are counted & no departures take place. Ex. ~~new~~ creation of birth certificate for newly born babies.

Let $P_n(t)$ be the probability of n arrivals in time t. & ~~λΔt~~ λ be average arrival ~~time~~ rate. Δt is small time interval which only one customer is arrive.

We make following assumptions:

(i) The prob. of an arrival in $(t, t+\Delta t)$ is $\lambda \Delta t$

(ii) The " " no " " $1-\lambda\Delta t$

(iii) The " more than one " " i.e time Δt is 0.

(iv) The number of arrivals in non overlapping intervals are statistically independent. i.e the process has independent arrivals.

Pure death process: Only departure are counted. No additional customer join the system while the service is continued for those who are in queue. μ be the service rate.

(i) Prob. of one departure in $(t, t+\Delta t)$ is $\mu\Delta t$

(ii) Prob. of no " " $1-\mu\Delta t$

(iii) prob. of more than one " " ~~intervals~~ is 0 (neg.)

(iv) the process has independent departures.

$$P_n(t+\Delta t) = P_{n+1}(t)\,\mu\Delta t + P_n(t)(1-\mu\Delta t)$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = (P_{n+1}(t) - P_n(t))\,\mu \longrightarrow (1).$$

SAMPLE SESSION

PCE	TUTORIAL SERIES	SHEET

DEPARTMENT: CLASS: SUB./CODE:

TOPIC............................ HOD:

NAME OF STUDENT.......................... SCHEDULED DATE OF SUBMISSION: DATE:........

FIRST 20 MT. CLASS QUESTIONS

for $n=0$

~~$P_0(t+\Delta t) = [P_1(t) - P_0(t)]\mu$~~

2 HRS SOLVABLE HOME WORK QUESTIONS

~~$P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t)$~~

at $n=N$ last, $P_{n+1}(t)=0$.

$\frac{dP_N(t)}{dt} = -\mu P_N(t) \Rightarrow \log P_N(t) = -\mu t + \log c_1$

$P_N(t) = c_1 e^{-\mu t}$ when $t=0$, $P_N=1$, $c_1=1$

When all customers have gone out by time t, then

$P_0(t+\Delta t) = 0 + P_1(t)\mu\Delta t$

$\frac{dP_0}{dt} = \mu P_1(t)$ when $n=0$.

$P_n(0) = \begin{cases} 1, & \text{when } n=N \\ 0, & \text{when } n \neq N \end{cases}$ are given by

OTHER IMPORTANT QUESTION

$P_n(t) = \frac{e^{-\mu t}(\mu t)^{N-n}}{(N-n)!}, \quad 1 \le n \le N$

Let $n=N-1$

$P'_{N-1}(t) = -\mu P_{N-1}(t) + \mu P_N(t)$

$P'_{N-1}(t) + \mu P_{N-1}(t) = \mu P_N(t)$

$= \mu e^{-\mu t}$

L.D. eqn.

I.F. $e^{\mu t}$

Soln. $e^{\mu t} \cdot P_{N-1}(t) = \int \mu e^{-\mu t} \cdot e^{\mu t} dt + c_2$

$= \mu t + c_2$

$P_{N-1}(t) = \mu t e^{-\mu t} + c_2 e^{-\mu t}$

at $t=0$, $P_{N-1}(t)=0$

$P_{N-1}(t) = \frac{\mu t}{\underline{1}} e^{-\mu t}$

For $n=N-2$

$P_{N-2}(t) = \frac{(\mu t)^2}{\underline{2}} e^{-\mu t}$

Similarly

$P_n(t) = \frac{(\mu t)^{N-n}}{\underline{N-n}} e^{-\mu t}$.

Also $\sum_{i=0}^{N}$ ...

...th-death process: When the customer arrivals corresponds to birth & customer departure after receiving service corresponds to death. (5)

Let λ_n & μ_n be the arrival rate (birth rate $n=0,1,2\ldots$) & departure (death rate) ($n=1,2\ldots$) the following assumption are satisfied. The system has n customer at time t.

$$P_n(t+\Delta t) = P_n(t)(1-\lambda_n\Delta t)(1-\mu_n\Delta t) + P_{n-1}(t)(\lambda_{n-1}\Delta t)(1-\mu_{n-1}\Delta t) + P_{n+1}(t)(1-\lambda_{n+1}\Delta t)(\mu_{n+1}\Delta t)$$

$$\lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -P_n(t)(\lambda_n+\mu_n) + P_{n-1}(t)\lambda_{n-1} + P_{n+1}(t)\mu_{n+1}$$

$$\Rightarrow P_n'(t) = \text{''} \rightarrow (1)$$

if $n=0$.

$$P_0'(t) = -P_0(t)(\lambda_0+\underset{0}{\underset{\|}{\mu_0}}) + 0 + P_1(t)\mu_1$$

no departure is possible when $n=0$.

$$P_0'(t) = -\lambda_0 P_0 + P_1\mu_1 \rightarrow (2).$$

Steady-state: $P_n(t)$ & $P_0(t)$ are independent of time

here

$$-P_n(t)(\lambda_n+\mu_n) + P_{n-1}(t)\cdot\lambda_{n-1} + P_{n+1}\mu_{n+1} = 0 \longrightarrow (3)$$

$$-\lambda_0 P_0 + P_1\mu_1 = 0 \longrightarrow (4).$$

$$P_1 = \frac{\lambda_0}{\mu_1}P_0 \rightarrow (5).$$

$n=1$ in (3).

$$\mu_2 P_2 = (\lambda_1+\mu_1)P_1(t) - P_0(t)\lambda_0.$$

$$= (\lambda_1+\mu_1)\frac{\lambda_0}{\mu_1}P_0 - P_0\lambda_0$$

$$\mu_2 P_2 = \lambda_0\lambda_1 P_0$$

$$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 \ldots$$

similarly.

$$P_n = \frac{\lambda_0\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}P_0, \quad n=1,2,\ldots$$

SAMPLE

SESSION: 2006-07

PCE 2006-07	TUTORIAL SERIES	SHEET#1
DEPARTMENT:	CLASS:	SUB./CODE:
TOPIC...........................		HOD:
NAME OF STUDENT........................	SCHEDULED DATE OF SUBMISSION:	DATE:.......

FIRST 20 MT. CLASS QUESTIONS

Since the number of customers in the system can be 0 or 1, 2, 3 — which events are mutually exclusive & exhaustive, we have $\sum_{n=0}^{\infty} P_n = 1$

$$P_0 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} P_0 = 1$$

2 HRS SOLVABLE HOME WORK QUESTIONS

$$\therefore P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right)}$$

The presence of n customers in the system at time $(t+\Delta t)$ can happen in any one of the following ans.

(i) presence of n customers ~~in the system~~ at t & no arrival & no departure in Δt.

(ii) presence of $(n-1)$ customers at t & one arrival & no departure in Δt

OTHER IMPORTANT QUESTION

(iii) presence of $(n+1)$ — " " & one departure & no arrival in Δt.

Steady state: If the characteristic of a queueing system are independent of time or if the behavior of the system is independent of time, the system is said to be in steady state. otherwise it is transitent state.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

$P_n(t)$ $\quad\quad \lambda$

$$P_n(t+\Delta t) = P_{n-1}\lambda\Delta t + P_n(t)(1-\lambda\Delta t)$$

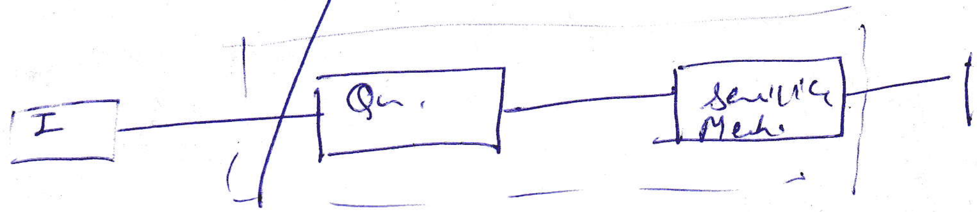
$$P_n(t+\Delta t) - P_n(t) = P_{n-1}(t)(\lambda\Delta t) - P_n(t)\lambda\Delta t$$

$$= \lambda\Delta t\{P_{n-1} - P_n\}$$

$$\lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \{P_{n-1}(t) - P_n(t)\}\lambda \quad , n \geq 1$$

$$\frac{dP_n}{dt} = \lambda\{P_{n-1}(t) - P_n(t)\}$$

for $n=0$, $P_{n-1}(t) = 0$



$$\cancel{P_n(t+\Delta t) = P_{n-1}(t)(\lambda\Delta t)(1-\mu\Delta t) + P_{n-1}(t)\lambda\Delta t(1+\mu\Delta t)}$$

$$P_n(t+\Delta t) = P_n(t)(1-\lambda\Delta t)(1-\mu\Delta t) + P_{n-1}(t)\lambda\Delta t(1-\mu\Delta t) + P_{n+1}(t)(1-\lambda\Delta t)(\mu\Delta t)$$

$\{\Delta^2 t \text{ neglect}\}$

$$\lim_{\Delta t\to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -P_n(t)(\lambda+\mu) + P_{n-1}(t)\lambda + P_{n+1}(t)\mu \longrightarrow (1)$$

$\frac{dP_n}{dt}$ $\quad\quad$ for $n \geq 1$.

for $n=0$, $P_{n-1} = 0$

$$\frac{dP_0}{dt} = -P_0(t)(\lambda+\mu) + P_1(t)\mu$$

$$P_0\mu = 0$$

$$\frac{dP_0}{dt} = -P_0\lambda + P_1\mu \longrightarrow (2)$$

In steady state condition are reached $\frac{dP_n}{dt} \to 0$

$$0 = -(\lambda+\mu)P_n + P_{n-1}\lambda + \mu P_{n+1} \quad , n \geq 1 \longrightarrow (3)$$

for $n=0$

$0 = -\lambda P_0 + \mu P_1 \longrightarrow$

since

$$P_0 + P_1 + - - - + P_\infty = 1$$

$$P_0 \left\{ 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + - - \right\} = 1$$

$\lambda/\mu < 1$

$$P_0 \frac{1}{1-\lambda/\mu} = 1 \Rightarrow \boxed{P_0 = 1 - \lambda/\mu = 1 - \rho} \rightarrow \text{at steady state}$$

Prob. of n customers at steddy state is

$$P_n = P_0 \left(\lambda/\mu \right)^n = (1-\rho)\,\rho^n.$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Birth death Process: where the customer arrivals corresponds of Birth & Customer departure after receiving service corresponds to death. We consider under the assumption of either no arrival or at most one arrival in the time interval with arrival rate λ in the time interval $(t, t+\Delta t)$ which true for poisson Queueing systems & either no departure or 1 departure with departure (death rate) μ in same intervals.

Let there be n customers present at time $(t+\Delta t)$ in the system with probability $P_n(t+\Delta t)$. we take following assumptions:

(i) presence of n customers at t & no arrival & no departure in Δt.

(ii) Presence of $(n-1)$ at t & 1 arrival & " " ".

(iii) Presence of $(n+1)$ " " & no arrival & 1 departure in Δt.

(iv) n customers present in time t & 1 arrival & 1 departure in Δt. Its prob. is $= P_n(t)\,\lambda\Delta t\,\mu\Delta t$

$= 0$ { $\Delta^2 t$ neglect terms. }

$$P_n(t+\Delta t) = P_{n-1}(t)(\lambda\Delta t)(1-\mu\Delta t) + P_n(t)(1-\mu\Delta t)(1-\lambda\Delta t) + P_{n+1}(t)(1-\lambda\Delta t)\mu\Delta t$$

$$\lim_{\Delta t\to 0}\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = \lambda P_{n-1}(t) - \mu P_n(t) - \lambda P_n(t) + P_{n+1}\mu$$

$$P_n'(t) = \frac{dP_n}{dt} = -(\lambda+\mu)P_n(t) + P_{n-1}(t)\lambda + P_{n+1}(t)\mu \longrightarrow (1)$$

for $n \geq 1$

for $n=0$, $P_{n-1}=0$

$$\frac{dP_0}{dt} = -P_0(t)(\lambda+\mu) + P_{n+1}(t)\mu$$

$P_0\mu = 0$:

$$\frac{dP_0}{dt} = -\lambda P_0 + P_1\mu \longrightarrow (2)$$

In steady state condition are reached $P_n' \to 0$

from (1) & (2).

~~$0 = -(\lambda+\mu)P_n(t) + P_{n-1}(t)\lambda + P_{n+1}(t)\mu$~~

$$-P_n(\lambda+\mu) + P_{n-1}\lambda + P_{n+1}\mu = 0 \quad, n\geq 1 \longrightarrow (3)$$

$n=1$ from (3)

$0 = -(\lambda+\mu)P_1 + P_0\lambda + P_2\mu$

$$P_2 = \left(\frac{\lambda+\mu}{\mu}\right)P_1 - P_0\lambda$$

$$= \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)P_0 - \frac{P_0\lambda}{\mu}$$

P_2 ...

$P_n =$

Classification of Queueing Models:

Queueing Model defined as $(A/B/C):(D/E)$

~~Here~~
Here A denotes arrival rate dis.
B ——— departure ———
C ——— Number of server
D ——— system Capacity
E denotes service discipline

Model I $(M/M/1):(\infty/FIFO)$: M denotes markovian property ~~Denotes~~ arrival & depature pattern after service both follows poission distribution, single server, infinite capacity & first come first served service discipline. & interarrival & service time distribution are exponential. (This model Based on Birth death process)

We derive the operational characteristic in the steady state taking the Process to be Birth Death process with arrival rate λ & service rate μ. Therefore ~~stat~~ steady state prob. is

$$P_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_n}{\mu_0 \mu_1 \cdots \mu_n} P_0$$

Taking $\lambda_n = \lambda$, $\mu_n = \mu$

$$P_n = P_0\left(\frac{\lambda}{\mu}\right)^n = P_0 \rho^n \quad , n = 0, 1, 2, \cdots \infty$$

we know that

$$P_0 + P_1 + \cdots = 1 \Rightarrow P_0\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots\right] = 1$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Charateristic of the Model:

(i) Probability of more than & equal to ~~a given number~~ n of customer to be present in the system.

$$P(\text{Number of customer in the system} \geq n) = \sum_{x=n}^{\infty} P_x$$

$$= \sum_{x=n}^{\infty} P_0 \left(\lambda/\mu\right)^x$$

$$= P_0 \left[\left(\lambda/\mu\right)^n + \left(\lambda/\mu\right)^{n+1} + \cdots\right]$$

$$= P_0 \left(\frac{\lambda}{\mu}\right)^n \left[1 + \lambda/\mu + \cdots\right]$$

$$= \frac{P_0 \left(\lambda/\mu\right)^n}{1 - \lambda/\mu}$$

$$\left\{\because P_0 = 1 - \lambda/\mu\right.$$

$$= \left(\lambda/\mu\right)^n = \rho^n$$

(ii) Average or mean number of customer present in the system:

$$L_s = E(n) = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n P_0 \left(\lambda/\mu\right)^n$$

$$= P_0 \left[0 + 1\left(\lambda/\mu\right) + 2\left(\lambda/\mu\right)^2 + \cdots\right]$$

$$= P_0 \left(\lambda/\mu\right) \left(1 - \lambda/\mu\right)$$

$$= P_0 \, \lambda/\mu \, \frac{1}{(1-\lambda/\mu)^2} \qquad \{\because P_0 = 1-\lambda/\mu\}$$

$$= \cancel{\lambda/\mu\,(1-\lambda/\mu)^{-1}} \; \frac{\lambda/\mu}{1-\lambda/\mu}$$

$$\boxed{L_s = E(n) = \frac{\lambda}{\mu-\lambda}}$$

(iii) Average number of customers in the queue (L_q)

$$L_q = E(m) = \sum_{m=0}^{\infty} m P_n \qquad \text{where } m = n-1$$

$$= \sum_{n=1}^{\infty} (n-1) P_n$$

$$= \sum_{n=1}^{\infty} n P_n - \sum_{n=1}^{\infty} P_n$$

$$= \frac{\lambda}{\mu-\lambda} - (1-P_0)$$

$$= \frac{\lambda}{\mu-\lambda} - (1-(1-\lambda/\mu))$$

$$= \frac{\lambda}{\mu-\lambda} - \lambda/\mu$$

$$\boxed{L_q = E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} \text{ or } E(n) - \lambda/\mu}$$

(iv) Probability that the number of customers in the system exceed K [we have to find $P(n>K)$]

$$= P_{K+1} + P_{K+2} + \text{---}$$

$$= \sum_{n=K+1}^{\infty} P_n = \sum_{n=K+1}^{\infty} P_0 (\lambda/\mu)^n = (\lambda/\mu)^{K+1} P_0 \left[1 + \lambda/\mu + (\lambda/\mu)^2 + \text{--}\right]$$

$$= (\lambda/\mu)^{K+1} \cancel{P_0} \cancel{(1-\lambda/\mu)^{-1}}$$

$$= (\lambda/\mu)^{K+1} = \rho^{K+1}$$

(V) Expected length of non-empty queues:

$$E\left(\frac{m}{m>0}\right) = E\left(\frac{n-1}{n-1>0}\right) = \frac{E(n-1)}{P(n-1>0)}$$

Now $P(n-1>0)$ ~~$P(\dots)$~~ $= P_2 + P_3 + \text{--}$

$$= P_0\left[(\lambda/\mu)^2 + (\lambda/\mu)^3 + \text{--}\right]$$

$$= P_0 (\lambda/\mu)^2 \left[1 + \lambda/\mu + \text{--}\right]$$

$$= P_0 (\lambda/\mu)^2 (1-\lambda/\mu)^{-1} \quad \left\{ \because P_0 = (1-\lambda/\mu) \right.$$

$$= (\lambda/\mu)^2 = \rho^2$$

so $E\left(\frac{m}{\dots}\right) = \frac{E(n-1)}{\dots} = \frac{\lambda^2}{\dots} \cdot (\mu/\dots)$

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Probability density $f(t)$ of waiting time in the system

$$f(t) = (\mu-\lambda)\, e^{-(\mu-\lambda)t}$$

Probability density for waiting time in the queue:

$$g(t) = \frac{\lambda}{\mu}(\mu-\lambda)\, e^{-(\mu-\lambda)t}, \quad t>0$$

for $t=0$

$$g(t) = \lambda/\mu\,(\mu-\lambda)$$

(i) ~~Characteristic~~ Average waiting time of a customer in the system:

$$E(\nu) = W_s = \int_0^\infty t f(t)\,dt = \int_0^\infty (\mu-\lambda)\, e^{-(\mu-\lambda)t}\, t\, dt$$

$$= (\mu-\lambda)\left[t\,\frac{e^{-(\mu-\lambda)t}}{-(\mu-\lambda)} - \frac{e^{-(\mu-\lambda)t}}{(\mu-\lambda)^2} \right]_0^\infty$$

$$= \frac{\mu-\lambda}{(\mu-\lambda)^2} = \frac{1}{\mu-\lambda}$$

$$\boxed{W_s = \frac{1}{\mu-\lambda}}$$

(ii) Average waiting time of a customer in the queue

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

(iii) Prob. that the waiting time a customer in the system exceeds t:

$$P(W_s > t) = \int_t^\infty f(t)\,dt$$

$$= \int_t^\infty (\mu-\lambda)e^{-(\mu-\lambda)t}\,dt$$

$$= (\mu-\lambda)\left[\frac{e^{-(\mu-\lambda)t}}{-(\mu-\lambda)}\right]_t^\infty = e^{-(\mu-\lambda)t}$$

Little formula : (Model I)

$$E(n) = L_s = \frac{\lambda}{\mu-\lambda}, \quad L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}, \quad W_s = \frac{1}{\mu-\lambda}, \quad W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$\boxed{L_s = \lambda W_s} \quad \& \quad \boxed{L_q = \lambda W_q}$$

$$\boxed{W_s = W_q + \frac{1}{\mu}}, \quad \boxed{L_s = L_q + \lambda/\mu}$$

DETAILED LECTURE NOTES

Campus: .... Course: .... Class/Section: .... Date: ....
Name of Faculty: .... Name of Subject: .... Code: ....

Q. On a telephone booth, arrivals of customers follow the Poison process with an average time of 10 minutes between one arrival and next arrival. The length of a phone call is assumed to be distibuted exponentially with mean 3 minutes.

(i) Find the average number of persons waiting in the system.

(ii) What is the Prob. that he spents more than 10 min. in the Booth?

(iii) What is the Prob. that a person arriving at the booth will have to wait?

(iv) Find the fraction of day's when the phone will be used.

Soln. (M/M/1) (∞/FCFS).

(i) ~~10min.~~ customer's arrives every 10 min. so

mean arrival rate $\lambda = \frac{1}{10} \times 60 = 6$ customer/hour

Mean service rate $\mu = 1/3 \times 60 = 20$ customers/hour.

(i) $L_s = \frac{\lambda}{\mu - \lambda} = \frac{6}{20-6} = 3/7$

(ii) $P(t > 10 \text{ min.}) = \int_t^\infty f(t)\,dt$ where $f(t) = (\mu-\lambda)\,e^{-(\mu-\lambda)t}$

$= \int_{10/60}^{\infty} (\mu-\lambda)\,e^{-(\mu-\lambda)t} = \cancel{(\mu-\lambda)}\left[\frac{e^{-(\mu-\lambda)t}}{-\cancel{(\mu-\lambda)}}\right]_{1/6}^{\infty}$

$= e^{-1/6(20-6)} = e$

DETAILED LECTURE NOTES

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(iii) P(person arriving at the booth will have to wait) :

$$= P_2 + P_3 + P_4 + \text{---}$$

$$= 1-(P_0+P_1)$$

$$= 1-\left(P_0 + P_0 \frac{\lambda}{\mu}\right)$$

$$\cancel{= \lambda/\mu - (1-\lambda/\mu - (1-\lambda/\mu)\lambda/\mu)}$$

$$= 1 - P_0(1+\lambda/\mu)$$

$$= 1-(1-\lambda/\mu)(1+\lambda/\mu)$$

$$= 1 - \left[1 - \cancel{\lambda/\mu} - \lambda/\mu + \frac{\lambda^2}{\mu^2}\right]$$

$$= \frac{\lambda^2}{\mu^2} = \frac{36}{400} = (.3)^2 = .09$$

(iv) Fraction of the day for which the phone is busy
= traffic intensity $= \lambda/\mu = .3 = 30\%$ of the day.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

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LECTURE NOTES

Campus: .... Course: .... Class/Section: .... Date: ....
Name of Faculty: Shilpi Jain Name of Subject: SPT Code: ....
Date (Prep.): .... Date (Del.): .... Unit No.: 5 Lect. No: ....

OBJECTIVE: To be written before taking the lecture (Pl. write in bullet points the main topics/concepts etc., which will be taught in this lecture)

→ Discrete-time Markov chains
→ Classification of states
→ Reversible Markov chains & detailed balance

IMPORTANT & RELEVANT QUESTIONS:

Numerical Question

FEED BACK QUESTIONS (AFTER 20 MINUTES):

What is Hitting time & absorption probabilities.
Detail of continuous time Markov chains.

OUTCOME OF THE DELIVERED LECTURE: To be written after taking the lecture (Pl. write in bullet points about students' feedback on this lecture, level of understanding of this lecture by students etc.)

students should be learn about the Discrete time Markov chains

REFERENCES: Text/Ref. Book with Page No. and relevant Internet Websites:

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

L-41.

1 Discrete-time Markov chains

1.1 Basic definitions and Chapman-Kolmogorov equation

(Very) short reminder on conditional probability. Let A, B, C be events.

* $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ (well defined only if $\mathbb{P}(B) > 0$)

* $\mathbb{P}(A \cap B|C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} \cdot \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} = \mathbb{P}(A|B \cap C) \cdot \mathbb{P}(B|C)$

Let now X be a discrete random variable.

* $\sum_k \mathbb{P}(X = x_k|B) = 1$

* $\mathbb{P}(B) = \sum_k \mathbb{P}(B|X = x_k)\, \mathbb{P}(X = x_k)$

* $\sum_k \mathbb{P}(X = x_k, A|B) = \mathbb{P}(A|B)$

but watch out that

* $\sum_k \mathbb{P}(B|X = x_k) \neq 1$

Definition 1.1. A *Markov chain* is a discrete-time stochastic process $(X_n, n \geq 0)$ such that each random variable X_n takes values in a discrete set S ($S = \mathbb{N}$, typically) and

$$\mathbb{P}(X_{n+1} = j \,|\, X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j|X_n = i)$$
$$\forall n \geq 0, j, i, i_{n-1}, \ldots, i_0 \in S$$

That is, as time goes by, the process loses the memory of the past.

If moreover $\mathbb{P}(X_{n+1} = j|X_n = i) = p_{ij}$ is independent of n, then X is said to be a *time-homogeneous* Markov chain. We will focus on such chains during the course.

Terminology.
* The possible values taken by the random variables X_n are called the *states* of the chain. S is called the *state space*.
* The chain is said to be *finite-state* if the set S is finite ($S = \{0, \ldots, N\}$, typically).
* $P = (p_{ij})_{i,j \in S}$ is called the *transition matrix* of the chain.

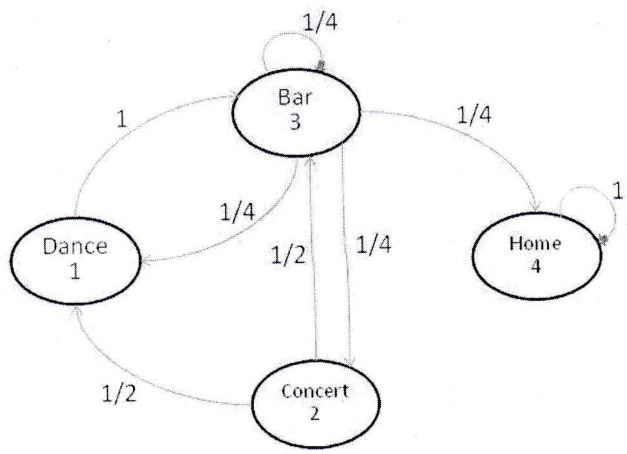
Properties of the transition matrix.
* $p_{ij} \geq 0, \forall i, j \in S$.
* $\sum_{j \in S} p_{ij} = 1, \forall i \in S$.

It is always possible to represent a time-homogeneous Markov chain by a transition graph.

Example 1.2. (music festival)
The four possible states of a student in a music festival are $S = \{$ "dancing", "at a concert", "at the bar", "back home"$\}$. Let us assume that the student changes state during the festival according to the following transition matrix:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This Markov chain can be represented by the following transition graph:



Example 1.3. (simple symmetric random walk)
Let $(X_n, n \geq 1)$ be i.i.d. random variables such that $\mathbb{P}(X_n = +1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$, and let $(S_n, n \geq 0)$ be defined as $S_0 = 0, S_n = X_1 + ... + X_n$, $n \geq 1$. Then $(S_n,\ n \in \mathbb{N})$ a Markov chain with state space $S = \mathbb{Z}$. Indeed:

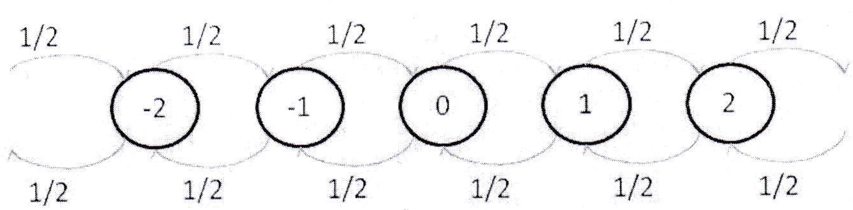
$$\mathbb{P}(S_{n+1} = j | S_n = i, S_{n-1} = i_{n-1}, ..., S_0 = i_0)$$
$$= \mathbb{P}(X_{n+1} = j - i | S_n = i, S_{n-1} = i_{n-1}, ..., S_0 = i_0) = \mathbb{P}(X_{n+1} = j - i)$$

by the assumption that the variables X_n are independent. The chain is moreover time-homogeneous, as

$$\mathbb{P}(X_{n+1} = j - i) = \begin{cases} \frac{1}{2} & \text{if } |j - i| = 1 \\ 0 & \text{otherwise} \end{cases}$$

does not depend on n.

Here is the transition graph of the chain:



The *distribution at time* n of the Markov chain X is given by:

$$\pi_i^{(n)} = \mathbb{P}(X_n = i), \quad i \in S$$

We know that $\pi_i^{(n)} \geq 0$ for all $i \in S$ and that $\sum_{i \in S} \pi_i^{(n)} = 1$.

The *initial distribution* of the chain is given by

$$\pi_i^{(0)} = \mathbb{P}(X_0 = i), \quad i \in S$$

It must be specified together with the transition matrix $P = (p_{ij}), i, j \in S$ in order to characterize the chain completely. Indeed, by repeatedly using the Markov property, we obtain:

$$\begin{aligned}
&\mathbb{P}(X_n = i_n, X_{n-1} = i_{n-1}, ..., X_1 = i_1, X_0 = i_0) \\
&= \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, ..., X_1 = i_1, X_0 = i_0) \cdot \mathbb{P}(X_{n-1} = i_{n-1}, ..., X_1 = i_1, X_0 = i_0) \\
&= p_{i_{n-1}, i_n}\, \mathbb{P}(X_{n-1} = i_{n-1}, ..., X_1 = i_1, X_0 = i_0) \\
&= \ldots = p_{i_{n-1}, i_n}\, p_{i_{n-2}, i_{n-1}} \cdots p_{i_1, i_2}\, p_{i_0, i_1}\, \pi_{i_0}^{(0)}
\end{aligned}$$

so knowing P and knowing $\pi^{(0)}$ allows to compute all the above probabilities, which give a compete description of the process.

The *n-step transition probabilities* of the chain are given by

$$p_{ij}^{(n)} = \mathbb{P}(X_{m+n} = j | X_m = i), \quad n, m \geq 0, \quad i, j \in S$$

Let us compute:

$$\begin{aligned}
p_{ij}^{(2)} &= \mathbb{P}(X_{n+2} = j | X_n = i) = \sum_{k \in S} \mathbb{P}(X_{n+2} = j, X_{n+1} = k | X_n = i) \\
&= \sum_{k \in S} \mathbb{P}(X_{n+2} = j | X_{n+1} = k, X_m = i) \cdot \mathbb{P}(X_{n+1} = k | X_n = i) \\
&= \sum_{k \in S} \mathbb{P}(X_{n+2} = j | X_{n+1} = k) \cdot \mathbb{P}(X_{n+1} = k | X_n = i) = \sum_{k \in S} p_{ik}\, p_{kj} \qquad (1)
\end{aligned}$$

where the Markov property property was used in (1). In a similar manner, we obtain the *Chapman-Kolmogorov equation* for generic values of m and n:

$$p_{ij}^{(n+m)} = \sum_{k \in S} p_{ik}^{(n)}\, p_{kj}^{(m)}, \quad i, j \in S, \quad n, m \geq 0$$

where we define by convention $p_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$

Notice that in terms of the transition matrix P, this equation simply reads:

$$(P^{n+m})_{ij} = (P^n\, P^m)_{ij} = \sum_{k \in S} (P^n)_{ik}\, (P^m)_{kj}$$

where, again by convention, $P^0 = I$, the identity matrix.

Notice also that

$$\pi_j^{(n)} = \mathbb{P}(X_n = j) = \sum_{i \in S} \mathbb{P}(X_n = j | X_{n-1} = i)\, \mathbb{P}(X_{n-1} = i) = \sum_{i \in S} p_{ij}\, \pi_i^{(n-1)}$$

In matrix form (considering $\pi^{(n)}$ as a row vector), this equation reads $\pi^{(n)} = \pi^{(n-1)} P$, from which we deduce that $\pi^{(n)} = \pi^{(n-2)} P^2 = \ldots = \pi^{(0)} P^n$, i.e.

$$\pi_j^{(n)} = \sum_{i \in S} p_{ij}^{(n)}\, \pi_i^{(0)}$$

1.2 Classification of states

We list here a set of basic definitions.

* A state j is *accessible* from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 0$.

* State i and j *communicate* if both j is accessible from i and i is accessible from j. Notation: $i \longleftrightarrow j$.

"To communicate" is an equivalence relation:

reflexivity: i always communicates with i (by definition).
symmetry: if i communicates with j, then j communicates with i (also by definition).
transitivity: if i communicates with j and j communicates with k, then i communicates with k (proof in the exercises)

* Two states that communicate are said to belong to the same *equivalence class*, and the state space S is divided into a certain number of such classes.

In Example 1.2, the state space S is divided into two classes: {"dancing", "at a concert", "at the bar"} and {"back home"}. In Example 1.3, there is only one class $S = \mathbb{Z}$.

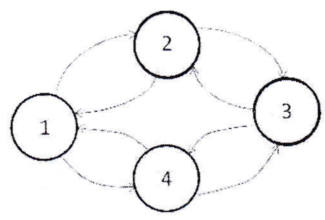
* The Markov chain is said to be *irreducible* if there is only one equivalence class (i.e. all states communicate with each other).

* A state i is *absorbing* if $p_{ii} = 1$.

* A state i is *periodic with period* d if d is the smallest integer such that $p_{ii}^{(n)} = 0$ for all n which are not multiples of d. In case $d = 1$, the state is said to be *aperiodic*.

* It can be shown that if a state i is periodic with period d, then all states in the same class are periodic with the same period d, in which case the whole class is periodic with period d.

Example 1.4. The Markov chain whose transition graph is given by



is an irreducible Markov chain, periodic with period 2.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

1.2.1 Recurrent and transient states

Let us recall here that $p_{ii}^{(n)} = \mathbb{P}(X_n = i | X_0 = i)$ is the probability, starting from state i, to come back to state i after n steps. Let us also define $f_i = \mathbb{P}(X \text{ever returns to } i | X_0 = i)$.

Definition 1.5. A state i is said to be *recurrent* if $f_i = 1$ or *transient* if $f_i < 1$.

It can be shown that all states in a given class are either recurrent or transient. In Example 1.2, the class {"dancing", "at a concert", "at the bar"} is transient (as there is a positive probability to leave the class and never come back) and the class {"back home"} is obviously recurrent. The random walk example 1.3 is more involved and requires the use of the following proposition.

Proposition 1.6.
* State i is recurrent if and only if $\sum_{n\geq 1} p_{ii}^{(n)} = \infty$.
* State i is transient if and only if $\sum_{n\geq 1} p_{ii}^{(n)} < \infty$.

Notice that the two lines of the above proposition are redundant, as a state is transient if and only if it is not recurrent.

Proof. Let T_i be the first time the chain X returns to state i. Therefore, $f_i = \mathbb{P}(T_i < \infty | X_0 = i)$. Let also N_i be the the number of times the chain X returns to state i and let us compute

$$\begin{aligned}\mathbb{P}(N_i < \infty | X_0 = i) &= \mathbb{P}(\exists n \geq 1 : X_n = i \;\&\; X_m \neq i,\ \forall m > n | X_0 = i)\\ &= \sum_{n\geq 1} \mathbb{P}(X_n = i \;\&\; X_m \neq i,\ \forall m > n | X_0 = i)\\ &= \sum_{n\geq 1} \mathbb{P}(X_m \neq i,\ \forall m > n | X_n = i, X_0 = i)\ \mathbb{P}(X_n = i | X_0 = i)\end{aligned}$$

As X is a time-homogeneous Markov chain, we have

$$\mathbb{P}(X_m \neq i,\ \forall m > n | X_n = i, X_0 = i) = \mathbb{P}(X_m \neq i,\ \forall m > n | X_n = i) = \mathbb{P}(X_m \neq i,\ \forall m > 0 | X_0 = i)$$

Therefore,

$$\begin{aligned}\mathbb{P}(N_i < \infty | X_0 = i) &= \mathbb{P}(X_m \neq i,\ \forall m > 0 | X_0 = i) \sum_{n\geq 1} \mathbb{P}(X_n = i | X_0 = i)\\ &= \mathbb{P}(T_i = \infty | X_0 = i) \sum_{n\geq 1} p_{ii}^{(n)} = (1 - f_i) \sum_{n\geq 1} p_{ii}^{(n)} \qquad (2)\end{aligned}$$

This implies that

- If i is recurrent, then $1 - f_i = 0$, so by (2), $\mathbb{P}(N_i < \infty | X_0 = i) = 0$, whatever the value of $\sum_{n\geq 1} p_{ii}^{(n)}$. This in turn implies

$$\mathbb{P}(N_i = \infty | X_0 = i) = 1, \quad \text{so} \quad \mathbb{E}(N_i | X_0 = i) = \infty$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

As $N_i = \sum_{n\geq 1} 1_{\{X_n=i\}}$, we also have

$$\infty = \mathbb{E}(N_i|X_0 = i) = \sum_{n\geq 1} \mathbb{E}(1_{\{X_1=i\}}|X_0 = i) = \sum_{n\geq 1} \mathbb{P}(X_n = i|X_0 = i) = \sum_{n\geq 1} p_{ii}^{(n)}$$

which proves the claim in this case.

- If on the contrary i is transient, then $1 - f_i > 0$ and as $\mathbb{P}(N_i = \infty|X_0 = i) \leq 1$, we obtain, using (2)

$$\sum_{n\geq 1} p_{ii}^{(n)} (1 - f_i) \leq 1 \quad \text{i.e.} \quad \sum_{n\geq 0} p_{ii}^{(n)} \leq \frac{1}{1 - f_i} < \infty$$

which completes the proof. □

Notice that as a by-product, we showed in this proof that if a state of a Markov chain is recurrent, then it is visited infinitely often by the chain, with probability 1 (therefore the name "recurrent").

Application. (simple random walk, symmetric or asymmetric)
Let us consider the simple random walk $(S_n,\ n \in \mathbb{N})$, with the following transition probabilities:

$$S_0 = 0, \quad \mathbb{P}(S_{n+1} = S_n + 1) = p = 1 - \mathbb{P}(S_{n+1} = S_n - 1) \quad \text{where } 0 < p < 1$$

Starting from 0, the probability to reach 0 after $2n$ steps is given by

$$p_{00}^{(2n)} = \mathbb{P}(S_{2n} = 0|S_0 = 0) = \binom{2n}{n} p^n (1 - p)^n, \quad \text{where} \quad \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

Notice that $p_{00}^{(2n+1)} = 0$ for all n and p, as an even number of steps is required to come back to 0. Using Stirling's approximation formula $n! \simeq n^n e^{-n} \sqrt{2\pi n}$, we obtain

$$p_{00}^{(2n)} \simeq \frac{(4p(1 - p))^n}{\sqrt{\pi n}}$$

From this expression, we see that if $p = 1/2$, then

$$\sum_{n\geq 1} p_{00}^{(n)} = \sum_{n\geq 1} p_{00}^{(2n)} = \sum_{n\geq 1} \frac{1}{\sqrt{\pi n}} = \infty$$

so by Proposition 1.6, state 0 is recurrent, and as the chain is irreducible, the whole chain is recurrent. If on the contrary $p \neq 1/2$, then $4p(1 - p) < 1$, so

$$\sum_{n\geq 1} p_{00}^{(n)} = \sum_{n\geq 1} p_{00}^{(2n)} = \sum_{n\geq 1} \frac{(4p(1 - p))^n}{\sqrt{\pi n}} < \infty$$

so state 0, and therefore the whole chain, is transient (in this case, the chain "escapes" to either $+\infty$ or $-\infty$, depending on the value of p).

Among recurrent states, we further make the following distinction (the justification for this distinction will come later).

Definition 1.7. Let i be a recurrent state and T_i be the first return time to state i.
* i is *positive recurrent* if $\mathbb{E}(T_i|X_0 = i) < \infty$
* i is *null recurrent* if $\mathbb{E}(T_i|X_0 = i) = \infty$

That is, if state i is null recurrent, then the chain comes back infinitely often to i, because the state is recurrent, but the time between two consecutive visits to i is on average infinite!

Notice that even if $f_i = \mathbb{P}(T_i < \infty|X_0 = i) = 1$, this does *not* imply that $\mathbb{E}(T_i|X_0 = i) < \infty$, as

$$\mathbb{E}(T_i|X_0 = i) = \sum_{n\geq 1} n\,\mathbb{P}(T_i = n|X_0 = i)$$

can be arbitrarily large.

Remarks.
* In a given class, all states are either positive recurrent, null recurrent or transient.
* In a finite state Markov chain, all recurrent states are actually positive recurrent.
* The simple symmetric random walk turns out to be null recurrent.

1.3 Stationary and limiting distributions

Let us first remember that a time-homogeneous Markov chain at time n is characterized by its distribution $\pi^{(n)} = (\pi_i^{(n)},\ i \in S)$, where $\pi_i^{(n)} = \mathbb{P}(X_n = i)$, and that

$$\pi^{(n+1)} = \pi^{(n)} P, \quad \text{i.e.} \quad \pi_j^{(n+1)} = \sum_{i\in S} \pi_i^{(n)}\, p_{ij}, \quad \forall j \in S$$

Definition 1.8. A distribution $\pi^* = (\pi_i^*,\ i \in S)$ is said to be a *stationary distribution* for the Markov chain $(X_n,\ n \geq 0)$ if

$$\pi^* = \pi^* P, \quad \text{i.e.} \quad \pi_j^* = \sum_{i\in S} \pi_i^*\, p_{ij}, \quad \forall j \in S \tag{3}$$

Remarks.
* π^* does not necessarily exist, nor is it necessarily unique.
* As we will see, if π^* exists and is unique, then π_i^* can always be interpreted as the average proportion of time spent by the chain X in state i. It also turns out in this case that

$$\mathbb{E}(T_i|X_0 = i) = \frac{1}{\pi_i^*}$$

where $T_i = \inf\{n \geq 0 : X_n = i\}$ is the first time the chain comes back to state i.
* If $\pi^{(0)} = \pi^*$, then $\pi^{(1)} = \pi^*\, P = \pi^*$; likewise, $\pi^{(n)} = \pi^*\, P^n = \ldots = \pi^*$, $\forall n \geq 0$, that is, if the initial distribution of the chain is stationary (we also say the chain is "in stationary state", by abuse of language), then it remains stationary over time.

Trivial example.
If $(X_n,\, n \geq 0)$ is a sequence of i.i.d. random variables, then $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_{n+1} = j)$ does actually depend neither on i nor on n , so $\pi_i^* = \mathbb{P}(X_n = i)$ (which is also independent of n) is a stationary distribution for the chain. Indeed,

$$\sum_{i \in S} \pi_i^* p_{ij} = \left(\sum_{i \in S} \pi_i^* \right) \mathbb{P}(X_n = j) = 1 \cdot \mathbb{P}(X_n = j) = \pi_j^*$$

Moreover, notice that in this example, $\pi^{(0)} = \pi^*$, so the chain is in stationary state from the beginning.

Definition 1.9. A distribution $\pi^* = (\pi_i^*,\, i \in S)$ is said to be a *limiting distribution* for the Markov chain $(X_n,\, n \geq 0)$ if for every initial distribution $\pi^{(0)}$ of the chain, we have

$$\lim_{n \to \infty} \pi_i^{(n)} = \pi_i^*, \quad \forall i \in S \tag{4}$$

Remarks.
* If π^* is a limiting distribution, then it is stationary. Indeed, for all $n \geq 0$, we always have $\pi^{(n+1)} = \pi^{(n)}\, P$. If $\lim_{n \to \infty} \pi^{(n)} = \pi^*$, then from the previous equation (and modulo a technical detail), we deduce that $\pi^* = \pi^* P$.
* A limiting distribution π^* does not necessarily exist, but if it exists, then it is unique.

The following theorem is central to the theory of Markov chains.

Theorem 1.10. Let $(X_n,\, n \geq 0)$ be an irreducible and aperiodic Markov chain. Let us moreover assume that it admits a stationary distribution π^*. Then π^* is a limiting distribution, i.e. for any initial distribution $\pi^{(0)}$, $\lim_{n \to \infty} \pi_i^{(n)} = \pi_i^*, \forall i \in S$.

We sketch the proof of this theorem below.

Proof. (sketch)
The idea behind the proof is the following *coupling argument*. Let $(X_n, n \geq 0)$ be the Markov chain decribed above and let $(Y_n,\, n \geq 0)$ be an independent replica of this one, except for the fact that Y starts with initial distribution π^* (so $\mathbb{P}(Y_n = i) = \pi_i^*$ for all $n \geq 0$ and all $i \in S$).

Let us now look at the bivariate process $(Z_n = (X_n, Y_n),\, n \geq 0)$. It can be shown that Z is also a Markov chain, with state space $S \times S$ and transition matrix

$$\mathbb{P}(Z_{n+1} = (j, l) | Z_n = (i, k)) = p_{ij}\, p_{kl}$$

As X and Y are both irreducible and aperiodic, Z is also irreducible and aperiodic. It also admits the following joint stationary distribution: $\Pi^*_{(i,k)} = \pi_i^* \pi_k^*$. We now use the following fact:

If a Markov chain is irreducible and admits a stationary distribution, then it is recurrent.

(This fact can be shown by contradiction: if an irreducible Markov chain is transient, then it cannot admit a stationary distribution.)

So Z is recurrent, which implies the following: let $\tau = \inf\{n \geq 0 : X_n = Y_n\}$ be the first time that the two chains X and Y meet. One can show that for all $n \geq 0$ and $i \in S$,

$$\mathbb{P}(X_n = i, \tau \leq n) = \mathbb{P}(Y_n = i, \tau \leq n)$$

But as Z is recurrent, we also have that $\mathbb{P}(\tau < \infty) = 1$, whatever the initial distribution $\pi^{(0)}$ of the chain X. So we obtain for $i \in S$:

$$\begin{aligned} |\pi_i^{(n)} - \pi_i^*| &= |\mathbb{P}(X_n = i) - \mathbb{P}(Y_n = i)| \\ &= |\mathbb{P}(X_n = i, \tau \leq n) - \mathbb{P}(Y_n = i, \tau \leq n)| + |\mathbb{P}(X_n = i, \tau > n) - \mathbb{P}(Y_n = i, \tau > n)| \\ &\leq 0 + \mathbb{P}(\tau > n) \underset{n\to\infty}{\to} 0 \end{aligned}$$

as $\mathbb{P}(\tau < \infty) = 1$. So π^* is a limiting distribution. □

Another equally important theorem is the following, However, its proof is more involved and will be skipped.

Theorem 1.11. Let $(X_n,\ n \geq 0)$ be an irreducible and positive recurrent Markov chain. Then X admits a unique stationary distribution π^*.

Remark.
An irreducible finite-state Markov chain is always positive recurrent. So by the above theorem, it always admits a unique stationary distribution.

Definition 1.12. A (time-homogeneous) Markov chain $(X_n,\ n \geq 0)$ is said to be *ergodic* if it is irreducible, aperiodic and positive recurrent.

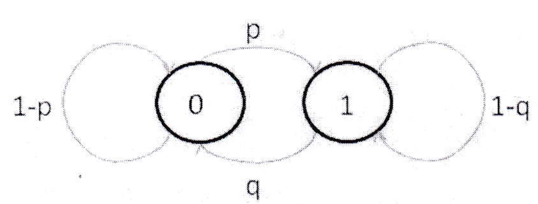
With this definition in hand, we obtain the following corollary of Theorems 1.10 and 1.11.

Corollary 1.13. An ergodic Markov chain $(X_n,\ n \geq 0)$ admits a unique stationary distribution π^*. Moreover, this distribution is also a limiting distribution, i.e.

$$\lim_{n\to\infty} \pi_i^{(n)} = \pi_i^*, \quad \forall i \in S$$

We give below a list of examples illustrating the previous theorems.

Example 1.14. (two-state Markov chain)
Let us consider a two-state Markov chain with the following transition graph (where $0 < p, q \leq 1$):



As both $p, q > 0$, this chain is clearly irreducible, and as it is finite-state, it is also positive recurrent. So by Theorem 1.11, it admits a stationary distribution. Writing down the equation for the stationary distribution $\pi = \pi P$, we obtain

$$\pi_0 = \pi_0 (1-p) + \pi_1 q, \quad \pi_1 = \pi_0 p + \pi_1 (1-q) \tag{5}$$

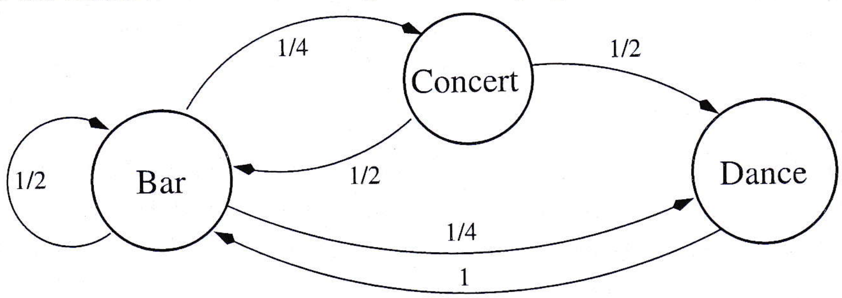
Remember also that as π is a distribution, we must have $\pi_0 + \pi_1 = 1$ and $\pi_0, \pi_1 \geq 0$. Solving these equations (and noticing that the two equations in (5) are actually redundant), we obtain

$$\pi_0 = \pi_0 (1-p) + (1-\pi_0) q \quad \Rightarrow \quad \pi_0 (p+q) = q, \quad \text{i.e.} \quad \pi_0 = \frac{q}{p+q}$$

so $\pi^* = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$ is the stationary distribution. Moreover, if $p+q < 2$ (i.e. if it is not the case that both $p = q = 1$), then the chain is also aperiodic and therefore ergodic, so by Corollary 1.13, $\pi^* = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$ is also a limiting distribution.

Notice that when both $p = q = 1$, then $\pi^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the unique stationary distribution of the chain, but in this case, the chain is periodic (with period $d = 2$) and π^* is *not* a limiting distribution. If for example the chain starts in state 0, then the distribution of the chain will switch from $\pi^{(n)} = (1, 0)$ at even times to $\pi^{(n)} = (0, 1)$ at odd times, and reciprocally, but it will never converge to the stationary distribution $\pi^* = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Example 1.15. (music festival: modified version)
Let us consider the chain with the following transition graph:



It has the corresponding transition matrix:

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

We can again easily see that the chain is ergodic. The computation of its stationary and limiting distribution gives

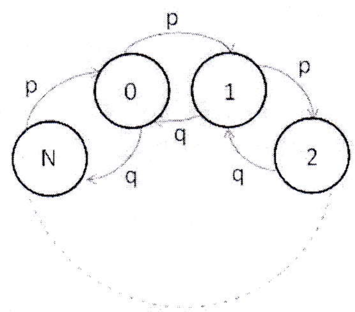
$$\pi^* = \left(\frac{8}{13}, \frac{2}{13}, \frac{3}{13}\right)$$

Quite unexpectedly, the student spends most of the time at the bar...

Example 1.16. (simple symmetric random walk)
Let us consider the simple symmetric random walk of Example 1.3. This chain is irreducible, periodic with period 2 and all states are null recurrent. There does not exist a stationary distribution here (NB: it should be the uniform distribution on $\mathbb{Z}$, which does not exist).

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Example 1.17. (cyclic random walk on $\{0, 1, \ldots, N\}$)
Let us consider the chain with the following transition graph (with $0 < p, q < 1$ and $p + q = 1$):



It has the corresponding transition matrix:

$$P = \begin{pmatrix} 0 & p & 0 & 0 & q \\ q & 0 & p & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & q & 0 & p \\ p & 0 & 0 & q & 0 \end{pmatrix}$$

This chain is irreducible and finite-state, so it is also positive recurrent, but its periodicity depends on the value of N: if N is odd (that is, the number of states is even), then the chain is periodic with period 2; if on the contrary N is even (that is, the number of states is odd), then the chain is aperiodic. In order to find its stationary distribution, observe that for all $j \in S$, $\sum_{i \in S} p_{ij} = p + q = 1$, so we can use Proposition 1.18 below to conclude that $\pi^* = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$, In case N is even, this distribution is also a limiting distribution.

Proposition 1.18. Let $(X_n, n \geq 0)$ be a finite-state irreducible Markov chain with state space $S = \{0, \ldots, N\}$ and let π^* be its unique stationary distribution (whose existence is ensured by Theorem 1.11 and the remark following it). Then π^* is the uniform distribution if and only if the transition matrix P of the chain satisfies:

$$\sum_{i \in S} p_{ij} = 1, \quad \forall j \in S$$

Remark.
Notice that the above condition is saying that the *columns* of the matrix P should sum up to 1, which is different from the condition seen at the beginning that the *rows* of the matrix P should sum up to 1 (satisfied by *any* transition matrix).

Proof. * If $\pi^* = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$ is a stationary distribution, then

$$\frac{1}{N+1} = \sum_{i \in S} \frac{1}{N+1} p_{ij}, \quad \forall j \in S, \quad \text{i.e.} \quad \sum_{i \in S} p_{ij} = 1, \quad \forall j \in S$$

* If $\sum_{i \in S} p_{ij} = 1, \forall j \in S$, then one can simply check that $\pi^* = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$ satisfies the equation $\pi^* = \pi^* P$. □

What happens without the aperiodicity assumption?

Theorem 1.19. Let $(X_n,\ n \geq 0)$ be an irreducible and positive recurrent Markov chain, and let π^* be its unique stationary distribution (whose existence is ensured by Theorem 1.11). Then for any initial distribution $\pi^{(0)}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \pi_i^{(k)} = \pi_i^*, \quad \forall i \in S$$

In this sense, π_i^* can still be interpreted as the average proportion of time spent by the chain in state i, and it also holds that

$$\mathbb{E}(T_i|X_0 = i) = \frac{1}{\pi_i^*}, \quad \forall i \in S$$

1.4 Reversible Markov chains and detailed balance

Let $(X_n,\ n \geq 0)$ be a time-homogeneous Markov chain. Let us now consider this chain *backwards*, i.e. consider the process $(X_n, X_{n-1}, X_{n-2}, \ldots, X_1, X_0)$: this process turns out to be also a Markov chain (but not necessarily time-homogeneous). Indeed:

$$\begin{aligned}
&\mathbb{P}(X_n = j|X_{n+1} = i, X_{n+2} = k, X_{n+3} = l, \ldots) \\
&= \frac{\mathbb{P}(X_n = j, X_{n+1} = i, X_{n+2} = k, X_{n+3} = l, \ldots)}{\mathbb{P}(X_{n+1} = i, X_{n+2} = k, X_{n+3} = l, \ldots)} \\
&= \frac{\mathbb{P}(X_{n+2} = k, X_{n+3} = l, \ldots, |X_{n+1} = i, X_n = j)}{\mathbb{P}(X_{n+2} = k, X_{n+3} = l, \ldots, |X_{n+1} = i)} \, \frac{\mathbb{P}(X_{n+1} = j, X_n = j)}{\mathbb{P}(X_{n+1} = i)} \\
&= \frac{\mathbb{P}(X_{n+2} = k, X_{n+3} = l, \ldots, |X_{n+1} = i)}{\mathbb{P}(X_{n+2} = k, X_{n+3} = l, \ldots, |X_{n+1} = i)} \, \mathbb{P}(X_n = j|X_{n+1} = i) \qquad (6) \\
&= \mathbb{P}(X_n = j|X_{n+1} = i)
\end{aligned}$$

where (6) follows from the Markov property of the forward chain X.

Let us now compute the transition probabilities:

$$\mathbb{P}(X_n = j|X_{n+1} = i) = \frac{\mathbb{P}(X_n = j, X_{n+1} = i)}{\mathbb{P}(X_{n+1} = i)} = \frac{\mathbb{P}(X_{n+1} = i|X_n = j)\,\mathbb{P}(X_n = j)}{\mathbb{P}(X_{n+1} = i)} = \frac{p_{ji}\,\pi_j^{(n)}}{\pi_i^{(n+1)}}$$

We observe that these transition probabilities may depend on n, so the backward chain is not necessarily time-homogeneous, as mentioned above.

Let us now assume that the chain is irreducible and positive recurrent. Then by Theorem 1.11, it admits a unique stationary distribution π^*. Let us moreover assume that the initial distribution of the chain is the stationary distribution (so the chain is in stationary state: $\pi^{(n)} = \pi^*$, $\forall n \geq 0$, i.e. $\mathbb{P}(X_n = i) = \pi_i^*$, $\forall n \geq 0$, $\forall i \in S$). In this case,

$$\mathbb{P}(X_n = j|X_{n+1} = i) = \frac{p_{ji}\,\pi_j^*}{\pi_i^*} = \widetilde{p}_{ij}$$

i.e. the backward chain is time-homogeneous with transition probabilities $\widetilde{p}_{ij}$.

Definition 1.20. The chain X is said to be *reversible* if $\widetilde{p}_{ij} = p_{ij}$, i.e. the transition probabilities of the forward and the backward chains are equal. In this case, the following *detailed balance equation* is satisfied:

$$\pi_i^* \, p_{ij} = \pi_j^* \, p_{ji}, \quad \forall i, j \in S \tag{7}$$

Remarks.

* If a distribution π^* satisfies the above detailed balance equation, then it is a stationary distribution. Indeed, if π^* satisfies (7), then

$$\sum_{i \in S} \pi_i^* \, p_{ij} = \sum_{i \in S} \pi_j^* \, p_{ji} = \pi_j^* \sum_{i \in S} p_{ji} = \pi_j^*, \quad \forall j \in S$$

* In order to find the stationary distribution of a chain, solving the detailed balance equation (7) is easier than solving the stationary distribution equation (3), but this works of course only if the chain is reversible.

* Equation (7) has the following interpretation: it says that in the Markov chain, the flow from state i to state j is equal to that from state j to state i.

* If equation (7) is satisfied, then π^* is the uniform distribution if and only if P is a symmetric matrix.

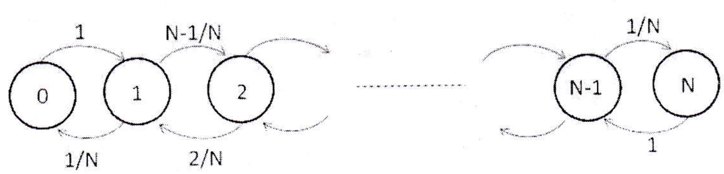
Example 1.21. (cyclic random walk on $\{0, 1, \ldots, N\}$)
Let us consider the cyclic random walk on $\{0, 1, \ldots, N\}$ of Example 1.17 with right and left transition probabilities p and q ($p + q = 1$). We have seen that the unique stationary distribution of this chain is the uniform distribution $\pi^* = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$. Is it the case that the detailed balance equation is satisfied here? By the above remark, this happens only when the transition matrix P is symmetric, i.e. when $p = q = 1/2$. Otherwise, we see that the flow of the Markov chain is more important in one direction than in the other.

Example 1.22. (Ehrenfest urns)
Let us consider the following process: there are 2 urns and N numbered balls. At each time step, one ball is picked uniformly at random among the N balls, and transferred from the urn where it lies to the other urn.

Let now X_n be the number of balls located in the first urn at time n. The process X describes a Markov chain on $\{0, \ldots, N\}$, whose transition probabilities are given by

$$p_{i,i+1} = \frac{N-i}{N} \quad p_{i,i-1} = \frac{i}{N} \quad \forall 1 \le i \le N-1 \quad \text{and} \quad p_{01} = 1, \quad p_{N,N-1} = 1$$

The corresponding transition graph is



Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

This chain is clearly irreducible, periodic with period 2 and positive recurrent, so it admits a unique stationary distribution π^*. A priori, we are not sure that the chain is reversible (although it is a reasonable guess in the present case), but we can still try solving the detailed balance equation and see where it leads:

$$\pi_i^* \, p_{i,i+1} = \pi_{i+1}^* p_{i+1,i} \quad \text{i.e.} \quad \pi_i^* \frac{N-i}{N} = \pi_{i+1}^* \frac{i+1}{N} \quad \Rightarrow \quad \pi_{i+1}^* = \frac{N-i}{i+1} \pi_i^*$$

So by induction, we obtain

$$\pi_i^* = \frac{(N-i+1)\cdots N}{i\cdots 1} \pi_0^* = \frac{N!}{(N-i)!\, i!} \pi_0^* = \binom{N}{i} \pi_0^*$$

Writing down the normalization condition $\sum_{i=0}^N \pi_i^* = 1$, we obtain

$$1 = \pi_0^* \sum_{i=0}^N \binom{N}{i} = \pi_0^* \, 2^N \quad \text{so} \quad \pi_i^* = 2^{-N} \binom{N}{i}, \qquad i = 0, 1, ..., N$$

Remark.
In physics, this process models the diffusion of particles across a porous membrane. It leads to the following paradox: assume the chain starts in state $X_0 = 0$ (that is, all the particles are on one side of the membrane), and let then the chain evolve over time. As the chain is recurrent, it will come back infinitely often to its initial state 0. This seems a priori in contradiction with the second principle of thermodynamics, which states that the entropy of a physical system should not decrease. Here, the entropy of the state 0 is much less than that of any other state in the middle, so the chain should not come back to 0 after having visited states in the middle. The paradox has been resolved by observing that for macroscopic systems (that is, $N \sim 6,022 \times 10^{23}$, the Avogadro number), the recurrence to state 0 is never observed in practice, as $\pi_0^* = 2^{-N}$.

1.5 Hitting times and absorption probabilities

Let $(X_n,\, n \geq 0)$ be a Markov chain with state space S and transition matrix P and let A be a subset of the state space S (notice that A need not be a class). In this section, we are interested in knowing what is the probability that the Markov chain X reaches a state in A. For this purpose, we introduce the following definitions.

Definition 1.23.
* *Hitting time*: $H_A = \inf\{n \geq 0 : X_n \in A\}$ = the first time the chain X "hits" the subset A.
* *Hitting probability*: $h_{iA} = \mathbb{P}(H_A < \infty | X_0 = i) = \mathbb{P}(\exists n \geq 0 \text{ such that } X_n \in A | X_0 = i)$, $i \in S$.

Remarks.
* The time H_A to hit a given set A might be infinity (if the chain never hits A).
* On the contrary, we say by convention that if $X_0 = i$ and $i \in A$, then $H_A = 0$ and $h_{iA} = 1$.
* If A is an absorbing set of states (i.e. there is no way for the chain to leave the set A once it has entered it), then the probability h_{iA} is called an *absorption probability*. A particular case that will be of interest to us is when A is a single absorbing state.

The following theorem allows to compute the vector of hitting probabilities $h_A = (h_{iA},\, i \in S)$.

Theorem 1.24. The vector $h_A = (h_{iA},\, i \in S)$ is the *minimal non-negative* solution of the following equation:

$$\begin{cases} h_{iA} = 1 & \forall i \in A \\ h_{iA} = \sum_{j\in S} p_{ij}\, h_{jA} & \forall i \notin A \end{cases} \tag{8}$$

By *minimal* solution, we mean that if $g_A = (g_{iA},\, i \in S)$ is another solution of (8), then $g_{iA} \geq h_{iA}, \forall i \in S$.

Remarks.
* The vector h_A is *not* a probability distribution, i.e. we do *not* have $\sum_{i\in S} h_{iA} = 1$.
* This theorem is nice, but notice that in order to compute a single hitting probability h_{iA}, one needs a priori to solve the equation for the entire vector h_A. It turns out however in many situations that solving the equation is much easier than computing directly hitting probabilities.

Proof. * Let us first prove that h_A is a solution of (8). If $i \in A$, then $h_{iA} = 1$, as $H_A = 0$ in this case. If $i \notin A$, then

$$\begin{aligned} h_{iA} &= \mathbb{P}(\exists n \geq 0 : X_n \in A | X_0 = i) = \mathbb{P}(\exists n \geq 1 : X_n \in A | X_0 = i) \\ &= \sum_{j\in S} \mathbb{P}(\exists n \geq 1 : X_n \in A | X_1 = j, X_0 = i)\, \mathbb{P}(X_1 = j | X_0 = i) \\ &= \sum_{j\in S} \mathbb{P}(\exists n \geq 1 : X_n \in A | X_1 = j)\, \mathbb{P}(X_1 = j | X_0 = i) \qquad (9) \\ &= \sum_{j\in S} \mathbb{P}(\exists n \geq 0 : X_n \in A | X_0 = j)\, p_{ij} = \sum_{j\in S} h_{jA}\, p_{ij} \qquad (10) \end{aligned}$$

where (9) follows from the Markov property and (10) follows from time-homogeneity.

Notice that if the state space S is finite, then it can be proved that there is a unique solution to equation (8), so the proof stops here.

* In general however, there might exist multiple solutions to equation (8). Let us then prove that h_A is minimal among these. For this purpose, assume g_A is another solution of (8). We want to prove that $g_{iA} \geq h_{iA}, \forall i \in S$. As g_A is a solution, we obtain the following.

If $i \in A$, then $g_{iA} = 1 = h_{iA}$. If $i \notin A$, then

$$\begin{aligned} g_{iA} &= \sum_{j\in S} p_{ij} g_{jA} = \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij} g_{jA} = \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij} \left(\sum_{k\in A} p_{jk} + \sum_{k\notin A} p_{jk}\, g_{kA} \right) \\ &= \mathbb{P}(X_1 \in A | X_0 = i) + \mathbb{P}(X_2 \in A, X_1 \notin A | X_0 = i) + \sum_{j\notin A} \sum_{k\notin A} p_{ij}\, p_{jk}\, g_{kA} \\ &= \mathbb{P}(X_1 \in A \text{ or } X_2 \in A | X_0 = i) + \sum_{j\notin A} \sum_{k\notin A} p_{ij}\, p_{jk}\, g_{kA} \end{aligned}$$

Observe that the last term on the right-hand side is non-negative, so

$$g_{iA} \geq \mathbb{P}(X_1 \in A \text{ or } X_2 \in A | X_0 = i)$$

This procedure can be iterated further and gives, for any $n \geq 1$:

$$g_{iA} \geq \mathbb{P}(X_1 \in A \text{ or } X_2 \in A \text{ or ... or } X_n \in A | X_0 = i)$$

So finally, we obtain

$$g_{iA} \geq \mathbb{P}(\exists n \geq 1 : X_n \in A | X_0 = i) = \mathbb{P}(\exists n \geq 0 : X_n \in A | X_0 = i) = h_{iA}$$

which completes the proof. □

We are also interested in knowing how long does the Markov chain X need to reach a state in A on average. For this purpose, let us introduce the following definition.

Definition 1.25. The *average hitting time* of a set A from a state $i \in S$ is defined as

$$\mu_{iA} = \mathbb{E}(H_A | X_0 = i) = \sum_{n \geq 0} n \, \mathbb{P}(H_A = n | X_0 = i)$$

Notice that this average hitting time might be ∞. The following theorem allows to compute the vector of average hitting times $\mu_A = (\mu_{iA}, \, i \in S)$. As its proof follows closely the one of Theorem 1.24, we do not repeat it here.

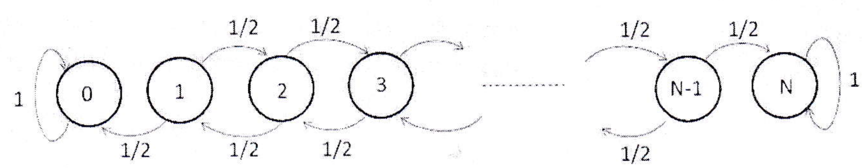
Theorem 1.26. The vector $\mu_A = (\mu_{iA}, i \in S)$ is the minimal non-negative solution of the following equation:

$$\begin{cases} \mu_{iA} = 0 & \forall i \in A \\ \mu_{iA} = 1 + \sum_{j \notin A} p_{ij} \, \mu_{jA} & \forall i \notin A \end{cases} \tag{11}$$

Please pay attention that this equation is similar to equation (8), but of course not identical.

We list below a series of examples where the above two theorems can be used.

Example 1.27. (gambler's ruin on $\{0, 1, 2, ..., N\}$)
Let us consider the time-homogeneous Markov chain with the following transition graph:



This chain models the following situation: a gambler plays "heads or tails" repeatedly, and each time wins or loses one euro with equal probability $1/2$; he plays until he either loses everything or wins a total amount of N euros. Assuming that he starts with i euros (with $1 \leq i \leq N-1$), what is the probability that he loses everything?

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

The answer is $h_{i0} = \mathbb{P}(H_0 < \infty | X_0 = i)$ (indeed, the only alternative is $H_N < \infty$). Let us therefore try solving equation (8):

$$\begin{cases} i = 0: & h_{00} = 1 \\ 1 \leq i \leq N-1: & h_{i0} = \frac{1}{2}(h_{i-1,0} + h_{i+1,0}) \quad \text{i.e.} \quad h_{i+1,0} = 2h_{i0} - h_{i-1,0} \\ i = N: & h_{N0} = 0 \end{cases} \tag{12}$$

Notice that there is actually no equation for h_{N0}; we therefore choose the smallest non-negative value, i.e. 0 (another view on this is that we know that $h_{N0} = 0$, as 0 is not accessilble from N). This gives

$$h_{20} = 2h_{10} - 1, \quad h_{30} = 2h_{20} - h_{10} = 3h_{10} - 2, \quad \ldots$$

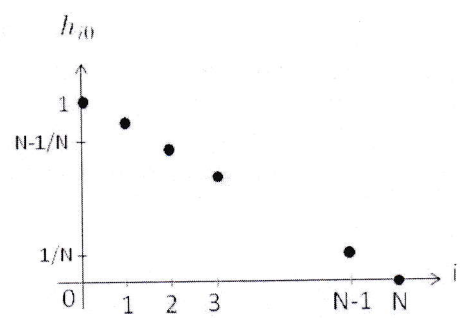
By induction, we obtain $h_{i0} = ih_{10} - (i-1)$, $\forall 1 \leq i \leq N-1$.

Writing down equation (12) for $i = N-1$, we further obatin

$$0 = h_{N0} = 2h_{N-1,0} - h_{N-2,0} = 2(N-1)h_{10} - 2(N-2) - (N-2)h_{10} + (N-3)$$

Therefore $Nh_{10} - N + 1 = 0$, i.e.

$$h_{10} = \frac{N-1}{N} \quad \text{and} \quad h_{i0} = i\frac{N-1}{N} - (i-1) = \frac{iN - i - Ni + N}{N} = \frac{N-i}{N}$$



Here is now a second question: how long will the game last on average (until the gambler either loses everything or wins N euros), assuming again the gambler starts with i euros ($1 \leq i \leq N-1$)?

The answer is the following: let us consider the subset $A = \{0, N\}$; the average duration of the game is $\mu_{iA} = \mathbb{E}(H_A | X_0 = i)$. Notice that $h_{iA} = 1$ (as there is no other alternative than to end in 0 or N) and also that the chain has a finite number of states, so $\mu_{iA} < \infty$ (whereas it can be checked that both $\mu_{i0} = \mu_{iN} = \infty$). The equation (11) for the vector μ_A reads in this case:

$$\begin{cases} i = 0: & \mu_{0A} = 0 \\ 1 \leq i \leq N-1: & \mu_{iA} = 1 + \frac{1}{2}(\mu_{i-1,A} + \mu_{i+1,A}) \quad \text{i.e.} \quad \mu_{i+1,A} = 2\mu_{iA} - 2 - \mu_{i-1,A} \\ i = N: & \mu_{NA} = 0 \end{cases} \tag{13}$$

The solution of this equation is obtained similarly to the previous one:

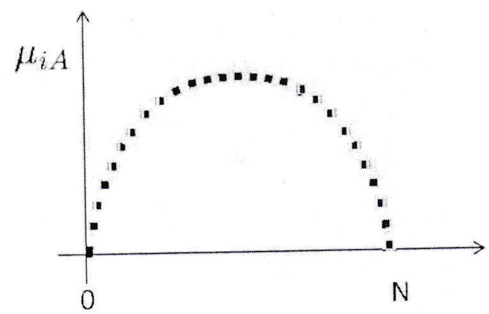
$$\mu_{2A} = 2\mu_{1A} - 2, \quad \mu_{3A} = 2\mu_{2A} - 2 - 2\mu_{1A} = 3\mu_{1A} - 6, \quad \ldots$$

so by induction, we obtain: $\mu_{iA} = i\mu_{1A} - i(i-1)$

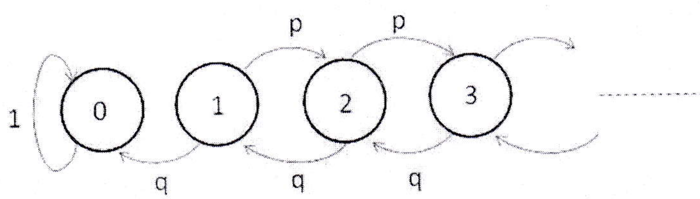
Writing down equation (13) for $i = N-1$, we further obtain

$$\begin{aligned} 0 = \mu_{NA} &= 2\mu_{N-1,A} - 2 - \mu_{N-2,A} \\ &= 2(N-1)\mu_{1A} - 2(N-1)(N-2) - 2 - (N-2)\mu_{1A} + (N-2)(N-3) \\ &= N\mu_{1A} - (N-2)(2(N-1) - (N-3)) - 2 \\ &= N\mu_{1A} - (N^2 - N - 2) - 2 = N\mu_{1A} - N^2 + N \end{aligned}$$

So $\mu_{1A} = \frac{N^2-N}{N} = N-1$ and $\mu_{iA} = i(N-1) - i(i-1) = i(N-i)$.



Example 1.28. (gambler's ruin on $\mathbb{N}$)
Let us consider the time-homogeneous Markov chain with the following transition graph:



This Markov chain describes a gambler playing repeatedly until he loses everything (there is no more upper limit N), winning each game with probability $0 < p < 1$ and losing with probability $q = 1-p$. Starting from a fortune of i euros, what is the probability that the gambler loses everything?

The answer is again h_{i0}, so let us try solving equation (8) (first assuming that $p \neq 1/2$):

$$h_{00} = 1, \quad h_{i0} = ph_{i+1,0} + qh_{i-1,0} \quad i \geq 1$$

The general solution of this difference equation is given by

$$h_{i0} = \alpha\, y_+^i + \beta\, y_-^i$$

where $y_\pm$ are the two solutions of the quadratic equation $y = py^2 + q$, i.e. $y_+ = 1$, $y_- = q/p$. Therefore,

$$h_{i0} = \alpha + \beta\,(q/p)^i$$

Using the boundary condition $h_{10} = ph_{20} + q$, we moreover obtain that $\alpha + \beta = 1$, i.e.

$$h_{i0} = \alpha + (1-\alpha)\,(q/p)^i$$

For any $\alpha \in [0, 1]$, the above expression is a non-negative solution of equation (8). The parameter α remains to be determined, using the fact that we are looking for the *minimal* solution.

* if $p < q$, then the minimal solution is given by $h_{i0} = 1, \forall i$ (i.e. $\alpha = 1$)

* If $p > q$, then the minimal solution is given by $h_{i0} = (q/p)^i, \forall i$ (i.e. $\alpha = 0$)

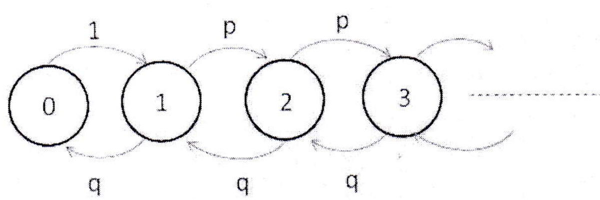
In the borderline case where $p = q = 1/2$, following what has been done in the previous example leads to

$$h_{i0} = ih_{10} - (i - 1), \quad \forall i$$

and we see that the minimal *non-negative* solution is actually given by $h_{10} = 1$, leading to $h_{i0} = 1$ for all i.

In conclusion, as soon as $p \leq 1/2$, the gambler is guaranteed to lose everything with probability 1, whatever his initial fortune.

Remarks.
* These absorption probabilities we just computed are also the hitting probabilities of the Markov chain with the following transition graph:



And for this chain, the probabilities $h_{10} = \mathbb{P}(H_0 < \infty | X_0 = 1)$ and $f_0 = \mathbb{P}(T_0 < \infty | X_0 = 0)$ are equal! So this new chain is recurrent if and only if $h_{10} = 1$, that is, if and only if $p \leq 1/2$.

* In the case $p = 1/2$, we can also compute the average hitting times μ_{i0}, following what has been done in the Example 1.27. We obtain:

$$\mu_{i0} = i\mu_{10} - i(i - 1), \quad \forall i$$

As $i(i - 1)$ increases faster to ∞ than i, we see that the vector μ_0 can be non-negative only if $\mu_{10} = \infty$ itself, i.e. $\mu_{i0} = \infty$ for all i. This is saying that in this case, the average time to reach 0 from any starting point i is actually infinite!

* Making now the connection between these two remarks, we see that for the chain described above, we have

$$\infty = \mu_{10} = \mathbb{E}(H_0 | X_0 = 1) = \mathbb{E}(T_0 | X_0 = 0)$$

i.e. the expected return time to state 0 is infinite, so the chain is null recurrent when $p = 1/2$ (similarly, it can be argued that the chain is positive recurrent when $p < 1/2$).

1.5.1 Application: branching processes

Here is a simple (not to say simplistic) model of evolution of the number of individuals in a population over the generations. Let first $(p_j,\ j \geq 0)$ be a given probability distribution.

Let now X_n describe the number of individuals in the population at generation n. At each generation n, each individual $i \in \{1, ..., X_n\}$ has C_i^n children, where $(C_i^n,\ i \geq 1,\ n \geq 0)$ are i.i.d. random variables with distribution $\mathbb{P}(C_i^n = j) = p_j$, $j \geq 0$. The number of individuals at generation $n+1$ is therefore:

$$X_{n+1} = C_1^n + ... + C_{X_n}^n$$

Because the random variable C_i^n are i.i.d., the process $(X_n,\ n \geq 0)$ is a time-homogeneous Markov chain (what happens to generation $n+1$ only depends on the value X_n, not on what happened before). Let us moreover assume that the population starts with $i > 0$ individuals.

We are interested in computing the extinction probability of this population, namely:

$$h_{i0} = \mathbb{P}(X_n = 0 \text{ for some } n \geq 1 | X_0 = i).$$

This model was originally introduced by Galton and Watson in the 19th century in order to study the extinction of surnames in noble families. It nowadays has found numerous applications in biology, and numerous variants of the model exist also.

Remarks.
* If $p_0 = \mathbb{P}(C_i^n = 0) = 0$, then the extinction probability $h_{i0} = 0$, trivially; let us therefore assume that $p_0 > 0$. In this case, 0 is an absorbing state and all the other states are transient.
* If a population starts with i individuals, then if extinction occurs, it has to occur for the family tree each of the i ancestors. So because of the i.i.d. assumption, the total extinction probability is the product of the extinction probabilities of each subtree, i.e. $h_{i0} = (h_{10})^i$.
* As a corollary, the fact that extinction occurs with probability 1 or not does not depend on the initial number of individuals in the population.
* For $i = 1$, the transition probabilities have the following simple expression:

$$p_{1j} = \mathbb{P}(X_{n+1} = j | X_n = 1) = \mathbb{P}(C_1^n = j) = p_j.$$

From Theorem 1.24, we know that the vector $h_0 = (h_{i0},\ i \geq 0)$ is the minimal non negative solution of

$$h_{00} = 1, \quad h_{i0} = \sum_{j \geq 0} p_{ij}\, h_{j0}, \quad i \geq 1$$

In particular, we obtain the following closed equation for h_{10}:

$$h_{10} = \sum_{j \geq 0} p_{1j}\, h_{j0} = \sum_{j \geq 0} p_j\, (h_{10})^j \tag{14}$$

In order to solve this equation for h_{10} (remembering that we are looking for the minimal non-negative solution), let us define the generating function

$$g(z) = \sum_{j \geq 0} p_j z^j, \quad z \in [0, 1]$$

Equation (14) can therefore be rewritten as the fixed point equation $h_{10} = g(h_{10})$. Its minimal non-negative solution is given by the following proposition.

Proposition 1.29. Let $\mu_c = \sum_{j\geq 1} p_j\, j$ be the average number of children of a given individual.

* If $\mu_c \leq 1$, then $h_{10} = 1$, i.e. extinction occurs with probability 1.

* If $\mu_c > 1$, then the minimal solution of $h_{10} = g(h_{10})$ is a number strictly between 0 and 1, so both extinction and survival occur with positive probability.

From this proposition, we see that slightly more than one child per individual is needed on average in order for the population to survive. But of course, there is always a positive probability that at some generation, no individual has a child and the population gets extinct.

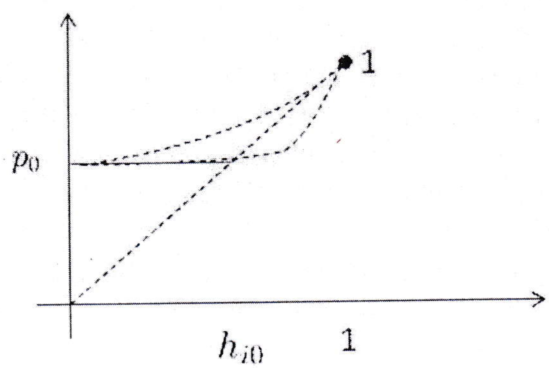
Proof. Let us analyze the properties of the generating function g:

* $g(0) = p_0 \in]0, 1]$, $g(1) = \sum_{j\geq 0} p_j = 1$

* $g'(z) = \sum_{j\geq 1} p_j\, j\, z^{i-1}$, so $g'(1) = \sum_{j\geq 1} p_j\, j = \mu_c$.

* $g''(z) = \sum_{j\geq 2} p_j\, j(j-1)\, z^{j-2} \geq 0$, so g is a convex function.

Given these properties, we see that only two things can happen:



* Either $\mu_c \leq 1$ (top curve), and then the unique solution to equation $h_{10} = g(h_{10})$ is $h_{10}^* = 1$.

* Or $\mu_c > 1$ (bottom curve), and then equation $h_{10} = g(h_{10})$ admits two solutions, the minimal of which is a number $h_{10}^* \in]0, 1[$. □

2 Continuous-time Markov chains

2.1 The Poisson process

Preliminary. (convergence of the binomial distribution towards the Poisson distribution)
Let $c > 0$ and $X_1, \ldots, X_M$ be i.i.d. random variables such that $\mathbb{P}(X_i = +1) = c/M$ and $\mathbb{P}(X_i = 0) = 1 - (c/M)$, for $1 \leq i \leq M$.

Let also $Z_M = X_1 + \ldots + X_M$. Then $Z_M \sim \text{Bi}(M, c/M)$, i.e.

$$\mathbb{P}(Z_M = k) = \binom{M}{k} (c/M)^k \, (1 - (c/M))^{M-k}, \quad 0 \leq k \leq M$$

Proposition 2.1. As $M \to \infty$, the distribution of Z_M converges to that of a Poisson random variable with parameter $c > 0$, i.e.

$$\mathbb{P}(Z_M = k) \underset{M\to\infty}{\to} \frac{c^k}{k!} \, e^{-k}, \quad \forall k \geq 0$$

Proof. Let us compute

$$\begin{aligned}
\mathbb{P}(Z_M = k) &= \binom{M}{k} (c/M)^k \, (1 - (c/M))^{M-k} \\
&= \frac{M(M-1)\cdots(M-k+1)}{k!} \, \frac{c^k}{M^k} \, (1 - (c/M))^M \, (1 - (c/M))^{-k} \underset{M\to\infty}{\to} \frac{c^k}{k!} \, e^{-c}
\end{aligned}$$

□

2.1.1 Definition and basic properties

The Poisson process is a continuous-time process counting events taking place at random times, such as e.g. customers arriving at a desk. Its definition follows.

Definition 2.2. A continuous-time stochastic process $(N_t, \, t \in \mathbb{R}_+)$ is a *Poisson process with intensity* $\lambda > 0$ if:

* N is integer-valued: $N_t \in \mathbb{N}$, $\forall t \in \mathbb{R}_+$.
* $N_0 = 0$ and N is increasing: $N_s \leq N_t$ if $s \leq t$.
* N has *independent and stationary increments*: for all $0 \leq t_1 \leq \ldots \leq t_m$ and $n_1, \ldots, n_m \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}(N_{t_1} = n_1, N_{t_2} - N_{t_1} = n_2, \ldots, N_{t_m} - N_{t_{m-1}} = n_m) \\
&\quad = \mathbb{P}(N_{t_1} = n_1) \, \mathbb{P}(N_{t_2} - N_{t_1} = n_2) \cdots \mathbb{P}(N_{t_m} - N_{t_{m-1}} = n_m)
\end{aligned}$$

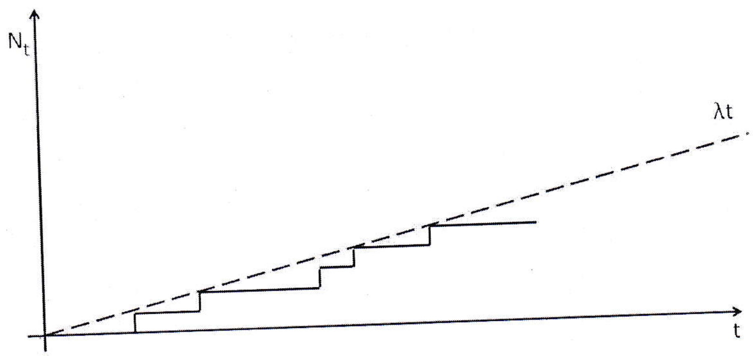
and for all $0 \leq s \leq t$ and $n, m \in \mathbb{N}$,

$$\mathbb{P}(N_t - N_s = n) = \mathbb{P}(N_{t-s} = n)$$

* $\mathbb{P}(N_{\Delta t} = 1) = \lambda \, \Delta t + o(\Delta t)$, $\mathbb{P}(N_{\Delta t} \geq 2) = o(\Delta t)$, where by definition $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.

NB: As a consequence of this definition, we see that $\mathbb{P}(N_{\Delta t} = 0) = (1 - \lambda \Delta t) + o(\Delta t)$.

Illustration. Here is a graphical representation of the time evolution of a Poisson process.



From the above definition, we deduce in the proposition below the distribution of the Poisson process at a given time instant.

Proposition 2.3. At time $t \in \mathbb{R}_+$, N_t is a Poisson random variable with parameter λt, i.e.

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k \geq 0$$

Proof. (sketch)
Let $t \in \mathbb{R}_+$, $M \geq 1$ and define $\Delta t = t/M$. We can write

$$N_t = \sum_{i=1}^{M} X_i, \quad \text{where} \quad X_i = N_{i\Delta t} - N_{(i-1)\Delta t}$$

From the last line of Definition 2.2 and the stationarity property, we deduce that

$$\mathbb{P}(X_i = 1) = \mathbb{P}(N_{\Delta t} = 1) \simeq \lambda \Delta t = \frac{\lambda t}{M}$$
$$\mathbb{P}(X_i \geq 2) = \mathbb{P}(N_{\Delta t} \geq 2) \simeq 0$$
$$\mathbb{P}(X_i = 0) = \mathbb{P}(N_{\Delta t} = 0) \simeq 1 - \lambda \Delta t = 1 - \frac{\lambda t}{M}$$

The random variables X_i can therefore be considered as (nearly) Bernoulli random variables with parameter $\frac{\lambda t}{M}$. Therefore,

$$\mathbb{P}(N_t = k) = \mathbb{P}(X_1 + \ldots + X_M = k) \simeq \binom{M}{k} \left(\frac{\lambda t}{M}\right)^k \left(1 - \frac{\lambda t}{M}\right)^{M-k}$$
$$\underset{M \to \infty}{\longrightarrow} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

by Proposition 2.1. □

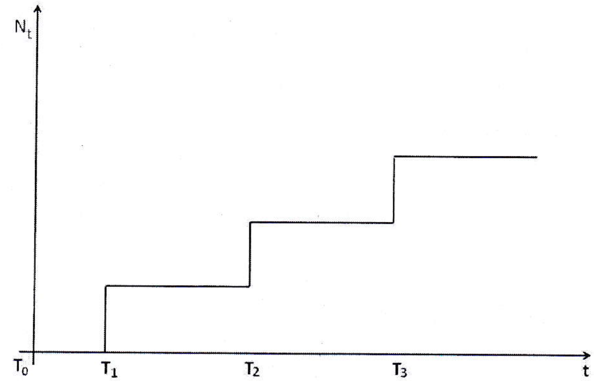
Corollary 2.4. For $t \in \mathbb{R}_+$, $\mathbb{E}(N_t) = \lambda t$ (so λ is the average number of events per unit time).

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

2.1.2 Joint distribution of the arrival times and inter-arrival times

Definition 2.5. The *arrival times* of a Poisson process are defined as

$$T_0 = 0, \quad T_n = \inf\{t \in \mathbb{R}_+ : N_t = n\}, \quad n \geq 1$$



The cumulative distribution function (cdf) of a given arrival time can be computed easily:

$$\mathbb{P}(T_n \leq t) = \mathbb{P}(N_t \geq n) = \sum_{k \geq n} \mathbb{P}(N_t = k) = \sum_{k \geq n} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

and its corresponding probability density function (pdf) is given by

$$\begin{aligned} p_{T_n}(t) &= \frac{d}{dt} \mathbb{P}(T_n \leq t) = \sum_{k \geq n} \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t} - \sum_{k \geq n} \frac{(\lambda t)^k}{k!} \lambda e^{-\lambda t} \\ &= \lambda e^{-\lambda t} \left(\sum_{k \geq n} \frac{(\lambda t)^{k-1}}{(k-1)!} - \sum_{k \geq n} \frac{(\lambda t)^k}{k!} \right) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

i.e. $T_n \sim \text{Gamma}(n, \lambda)$ (and remember that such a Gamma random variable can be written as the sum of n i.i.d. exponential random variables, each of parameter $\lambda > 0$),

We now would like to compute the *joint* distribution of the arrival times $T_1, \ldots, T_n$. For this, let us recall the following.

* If T is a non-negative random variable, then for $0 \leq a < b$, $\mathbb{P}(a < T \leq b) = \int_a^b dt \, p_T(t)$, where p_T is the pdf of T.

* Similarly, if $T_n \geq \ldots \geq T_2 \geq T_1$ are non-negative random variables, then for $0 \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n$, we have

$$\begin{aligned} &\mathbb{P}(a_1 < T_1 \leq b_1, \, a_2 < T_2 \leq b_2, \ldots, \, a_n < T_n \leq b_n) \\ &= \int_{a_1}^{b_1} dt_1 \int_{a_2}^{b_2} dt_2 \cdots \int_{a_n}^{b_n} dt_n \, p_{T_1,\ldots,T_n}(t_1, \ldots, t_n) \end{aligned}$$

where $p_{T_1,\ldots,T_n}$ is the joint pdf of $T_1, \ldots, T_n$.

Let us therefore compute

$$\mathbb{P}(a_1 < T_1 \le b_1,\, a_2 < T_2 \le b_2, \ldots, a_n < T_n \le b_n)$$

$$= \mathbb{P}(N_{a_1} = 0,\, N_{b_1} - N_{a_1} = 1,\, N_{a_2} - N_{b_1} = 0, \ldots,\, N_{a_n} - N_{b_{n-1}} = 0,\, N_{b_n} - N_{a_n} \ge 1)$$

$$= \mathbb{P}(N_{a_1} = 0)\,\mathbb{P}(N_{b_1} - N_{a_1} = 1)\,\mathbb{P}(N_{a_2} - N_{b_1} = 0)\cdots\mathbb{P}(N_{a_n} - N_{b_{n-1}} = 0)\,\mathbb{P}(N_{b_n} - N_{a_n} \ge 1)$$

$$= e^{-\lambda a_1}\,\lambda(b_1 - a_1)\,e^{-\lambda(b_1 - a_1)}\,e^{-\lambda(a_2 - b_1)}\cdots e^{-\lambda(a_n - b_{n-1})}\,\left(1 - e^{-\lambda(b_n - a_n)}\right)$$

$$= \lambda^{n-1}\prod_{i=1}^{n-1}(b_i - a_i)\,(e^{-\lambda a_n} - e^{-\lambda b_n}) = \int_{a_1}^{b_1} dt_1 \cdots \int_{a_n}^{b_n} dt_n\, \lambda^n\, e^{-\lambda t_n}$$

So the joinf pdf of $T_1, \ldots, T_n$ is given by

$$p_{T_1,\ldots,T_n}(t_1, \ldots, t_n) = \lambda^n\, e^{-\lambda t_n}\, 1_{\{0 \le t_1 \le \ldots \le t_n\}}$$

In particular,

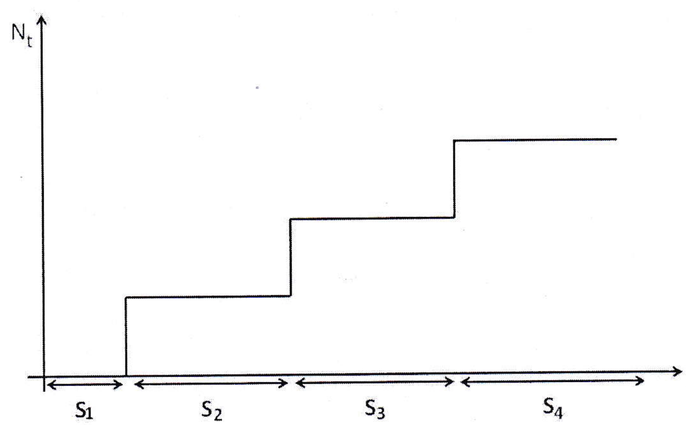
$$\begin{aligned} p_{T_1,\ldots,T_{n-1}|T_n}(t_1, \ldots, t_{n-1}|t_n) &= \frac{p_{T_1,\ldots,T_n}(t_1, \ldots, t_n)}{p_{T_n}(t_n)} \\ &= \frac{\lambda^n e^{-\lambda t_n}}{\lambda^n (t_n)^{n-1}\, e^{-\lambda t_n}/(n-1)!}\, 1_{\{0 \le t_1 \le \ldots \le t_n\}} = \frac{(n-1)!}{(t_n)^{n-1}}\, 1_{\{0 \le t_1 \le \ldots \le t_n\}} \end{aligned}$$

i.e., given that $T_n = t_n$, the random variables $T_1, \ldots, T_{n-1}$ have the same distribution as the order statistics of $n-1$ random variables uniformly distributed on $[0, t_n]$.

Definition 2.6. The *inter-arrival times* of a Poisson process are defined as

$$S_n = T_n - T_{n-1}, \quad n \ge 1$$

Equivalently, $T_n = S_1 + S_2 + \ldots + S_n$, for $n \ge 1$.



The joint pdf of $S_1, \ldots, S_n$ can be easily computed from the joint pdf of $T_1, \ldots, T_n$:

$$p_{S_1,\ldots,S_n}(s_1, \ldots, s_n) = p_{T_1,\ldots,T_n}(s_1, s_1 + s_2, \ldots, s_1 + \ldots + s_n)$$

$$= \lambda^n e^{-\lambda(s_1+\ldots+s_n)} 1_{\{s_1 \geq 0, s_1+s_2 \geq s_1, \ldots, s_1+\ldots+s_n \geq s_1+\ldots+s_{n-1}\}} = \prod_{i=1}^{n} \lambda e^{-\lambda s_i} 1_{s_i \geq 0}$$

i.e. $S_1, \ldots, S_n$ are n i.i.d. exponential random variables with parameter λ (and as already observed above, T_n is the sum of them). This gives rise to the following proposition (which can also be taken as an alternate definition of the Poisson process).

Proposition 2.7. Let $(S_n,\ n \geq 1)$ be i.i.d. exponential random variables with parameter $\lambda > 0$. Then the process defined as

$$N_t = \max\{n \geq 0 : S_1 + \ldots + S_n \leq t\}, \quad t \in \mathbb{R}_+$$

is a Poisson process of intensity $\lambda > 0$.

Remark.
The exponential distribution of the inter-arrival times leads to the following consequence:
* Let $t_0 \in \mathbb{R}_+$ be a fixed time, chosen independently of the process N. Then by stationarity,

$$\mathbb{P}(N_{t_0+t} - N_{t_0} \geq 1) = \mathbb{P}(N_t \geq 1) = 1 - \mathbb{P}(N_t = 0) = 1 - e^{-\lambda t}$$

* Let us now replace t_0 by an arrival time of the process T_n. Then again,

$\mathbb{P}(N_{T_n+t} - N_{T_n} \geq 1) = \mathbb{P}(S_{n+1} \leq t) = 1 - e^{-\lambda t}$, i.e. the probability is the same as before!

So the probability that an event takes place t seconds after a given time does not depend on whether this given time is an arrival time of the process or not.

2.1.3 Additional properties

We prove below two useful propositions.

Proposition 2.8. (superposition of two independent Poisson processes)
Let $N^{(1)}$, $N^{(2)}$ be two independent Poisson processes with intensity λ_1 and λ_2, respectively. Then the process N defined as

$$N_t = N_t^{(1)} + N_t^{(2)}, \quad t \in \mathbb{R}_+$$

is again a Poisson process, with intensity $\lambda_1 + \lambda_2$.

Proof. (sketch)
We only prove here that for $t \in \mathbb{R}_+$, N_t is a Poisson random variable with parameter $(\lambda_1 + \lambda_2)\, t$:

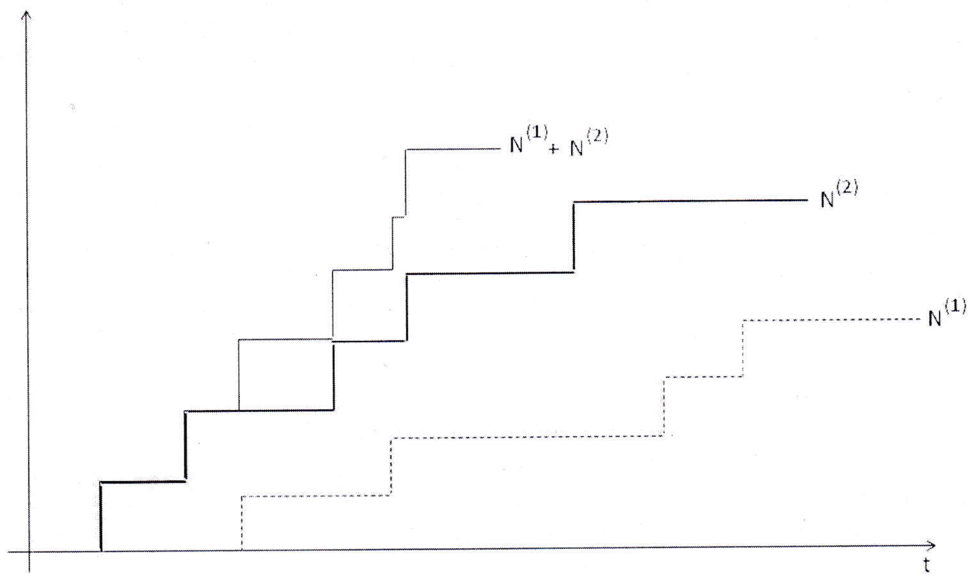
$$\mathbb{P}(N_t = n) = \mathbb{P}(N_t^{(1)} + N_t^{(2)} = n) = \sum_{k=0}^{n} \mathbb{P}(N_t^{(1)} = k, N_t^{(2)} = n - k)$$

$$= \sum_{k=0}^{n} \mathbb{P}(N_t^{(1)} = k)\, \mathbb{P}(N_t^{(2)} = n - k) = \sum_{k=0}^{n} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{n-k}}{(n-k)!} e^{-\lambda_2 t}$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} \frac{t^n}{n!} e^{-(\lambda_1+\lambda_2)t} = \frac{((\lambda_1 + \lambda_2)t)^n}{n!} e^{-(\lambda_1+\lambda_2)t}$$

□

The superposition of two Poisson processes is illustrated on the figure below.



The next proposition is in some sense the reciprocal of the former one.

Proposition 2.9. (thinning of a Poisson process)
Let N be a Poisson process with intensity λ and let $(X_n,\ n \geq 1)$ be a sequence of i.i.d. random variables independent of N and such that $\mathbb{P}(X_n = 1) = p = 1 - \mathbb{P}(X_n = 0)$, with $0 < p < 1$. Let us then associate to each arrival time T_n of the original process N a random variable X_n and let $N^{(1)}$ be the process whose arrival times are those of the process N for which $X_n = 1$. Then $N^{(1)}$ is again a Poisson process, with intensity $p\lambda$.

Proof. (sketch)
We only prove again that for $t \in \mathbb{R}_+$, $N_t^{(1)}$ is a Poisson random variable with parameter $p\lambda t$:

$$\begin{aligned}
\mathbb{P}(N_t^{(1)} = k) &= \sum_{n \geq k} \mathbb{P}(N_t = n, X_1 + \ldots + X_n = k) \\
&= \sum_{n \geq k} \mathbb{P}(N_t = n)\, \mathbb{P}(X_1 + \ldots + X_n = k) \qquad (15) \\
&= \sum_{n \geq k} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \\
&= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \sum_{n \geq k} \frac{(\lambda t)^{n-k}}{(n-k)!} (1-p)^{n-k} = \frac{(p\lambda t)^k}{k!} e^{-p\lambda t}
\end{aligned}$$

where (15) follows from the assumed independence between the process N and the random variables $(X_n,\ n \geq 1)$. □

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Classification of Markov chain

2.2 Continuous-time Markov chains

2.2.1 Definition and basic properties

Definition 2.10. A *continuous-time Markov chain* is a stochastic process $(X_t,\ t \in \mathbb{R}_+)$ with values in a discrete state space S such that

$$\mathbb{P}(X_{t_{n+1}} = j | X_{t_n} = i, X_{t_{n-1}} = i_{n-1}, \ldots, X_{t_0} = i_0) = \mathbb{P}(X_{t_{n+1}} = j | X_{t_n} = i),$$
$$\forall j, i, i_{n-1}, \ldots, i_0 \in S \quad \text{and} \quad \forall t_n > t_{n-1} > \ldots > t_0 \geq 0$$

If moreover

$$\mathbb{P}(X_{t+s} = j | X_s = i) = \mathbb{P}(X_t = j | X_0 = i) = p_{ij}(t), \quad \forall i, j \in S \quad \text{and} \quad \forall t, s \geq 0$$

then the chain is said to be *time-homogeneous* (we only consider such chains in the following).

Notice that we do not have anymore a single transition matrix $P = (p_{ij})_{i,j \in S}$, but a *collection* of transition matrices $P(t) = (p_{ij}(t))_{i,j \in S}$, indexed by $t \in \mathbb{R}_+$.

Example 2.11. The Poisson process with intensity $\lambda > 0$ is a continuous-time Markov chain, Indeed, for $j \geq i \geq i_{n-1} \geq \ldots \geq i_0 \in \mathbb{N}$, we have:

$$\begin{aligned}
&\mathbb{P}(N_{t_{n+1}} = j | N_{t_n} = i, N_{t_{n-1}} = i_{n-1}, \ldots, N_{t_0} = i_0) \\
&\quad = \mathbb{P}(N_{t_{n+1}} - N_{t_n} = j - i | N_{t_n} = i, N_{t_{n-1}} = i_{n-1}, \ldots, N_{t_0} = i_0) \\
&\quad = \mathbb{P}(N_{t_{n+1}} - N_{t_n} = j - i) = \mathbb{P}(N_{t_{n+1} - t_n} = j - i)
\end{aligned}$$

where the last line follows from the independence and the stationarity of increments. Similarly, we obtain

$$\mathbb{P}(N_{t_{n+1}} = j | N_{t_n} = i) = \mathbb{P}(N_{t_{n+1} - t_n} = j - i)$$

proving therefore the Markov property. Furthermore, the transition probabilities are given by

$$p_{ij}(t) = \mathbb{P}(N_t = j - i) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}$$

Notations.
* $\pi(t) = (\pi_i(t), i \in S)$ is the *distribution of the Markov chain at time* $t \in \mathbb{R}_+$.
i.e. $\pi_i(t) = \mathbb{P}(X_t = i)$. Again, we have $\pi_i(t) \geq 0$ for all $i \in S$ and $\sum_{i \in S} \pi_i(t) = 1$, $\forall t \in \mathbb{R}_+$.
* $\pi(0) = (\pi_i(0), i \in S)$ is the *initial distribution* of the Markov chain.

One can check that $\pi_j(t) = \sum_{i \in S} \pi_i(0)\, p_{ij}(t)$ and $\pi_i(t+s) = \sum_{i \in S} \pi_i(t)\, p_{ij}(s)$.

The *Chapman-Kolmogorov equation* reads in the continuous-time case as

$$p_{ij}(t+s) = \sum_{k \in S} p_{ik}(t)\, p_{kj}(s), \quad \forall i, j \in S, \quad t, s \in \mathbb{R}_+$$

Proof. Along the lines of what has been shown in the discrete-time case, we obtain

$$\begin{aligned}
p_{ij}(t+s) &= \mathbb{P}(X_{t+s} = j | X_0 = i) = \sum_{k \in S} \mathbb{P}(X_{t+s} = j, X_t = k | X_0 = i) \\
&= \sum_{k \in S} \mathbb{P}(X_{t+s} = j | X_t = k, X_0 = i)\, \mathbb{P}(X_t = k | X_0 = i) = \sum_{k \in S} p_{ik}(t)\, p_{kj}(s)
\end{aligned}$$

□

2.2.2 Transition and sojourn times

Disclaimer. In this section and the following ones, rigorous proofs are often missing!

Definition 2.12. The *transition and sojourn times* of a continuous-time Markov chain are defined respectively as

$$T_0 = 0, \quad T_{n+1} = \inf\{t > T_n : X_t \neq X_{T_n}\}, \quad n \geq 0$$

and

$$S_n = T_n - T_{n-1}, \quad n \geq 1$$

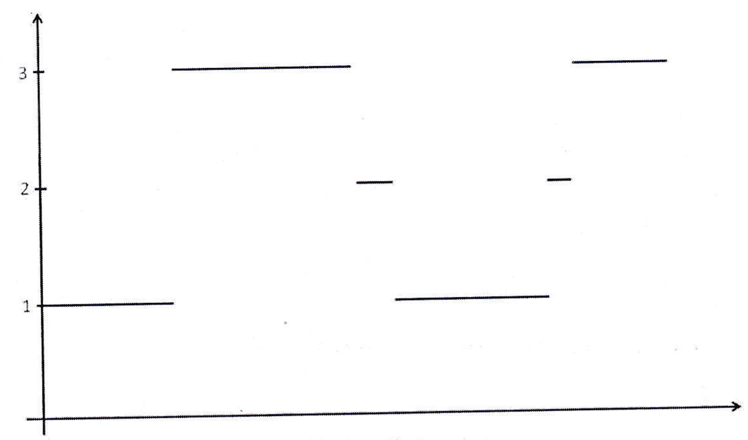
Equivalently, $T_n = S_1 + \ldots + S_n$.

The following fact is essentially a consequence of the Markov property.

Proposition 2.13. (without proof)
The sojourn times $(S_n,\, n \geq 1)$ are independent exponential random variables.

Remarks.
* In general, the sojourn times $(S_n,\, n \geq 1)$ are not identically distributed random variables.
* Also, the parameter of the exponential random variable S_{n+1} depends on the state of the Markov chain at time T_n. That is, given that $X_{T_n} = i$, S_{n+1} is an exponential random variable with some parameter ν_i, so that the average waiting time in state i is $\mathbb{E}(S_{n+1}|X_{T_n} = i) = 1/\nu_i$. The parameter ν_i is the rate at which the chain leaves state i.

Here is the graphical representation of the time evolution of a Markov chain with 3 states and $\nu_1 \sim \nu_3 < \nu_2$:



$\nu_1, \nu_3 < \nu_2$

In order to avoid strange behaviors (such as e.g. processes with infinitely many transitions during a fixed period of time), we make the following *additional assumption*:

$$\sum_{i=1}^{n} S_i = T_n \underset{n\to\infty}{\to} \infty$$

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

2.2.3 Embedded discrete-time Markov chain

Let us define $\widehat{X}_n = X_{T_n}$ and $\widehat{q}_{ij} = \mathbb{P}(X_{T_{n+1}} = j | X_{T_n} = i) = \mathbb{P}(\widehat{X}_{n+1} = j | \widehat{X}_n = i)$, $i, j \in S$.

Fact. (without proof)
The process $(\widehat{X}_n,\ n \geq 0)$ is a discrete-time Markov chain with transition probabilities $(\widehat{q}_{ij})_{i,j \in S}$. It is said to be *embedded* in the continuous-time Markov chain $(X_t,\ t \in \mathbb{R}_+)$.

Remark.
* Notice indeed that $\widehat{q}_{ij} \geq 0$, $\forall i, j \in S$ and $\sum_{j \in S} \widehat{q}_{ij} = 1$, $\forall i \in S$, as in the discrete-time case.
* Here, in addition, $\widehat{q}_{ii} = 0$, $\forall i \in S$, i.e. the embedded discrete-time Markov chain never has self-loops in its transition graph.
* The embedded chain does not "see" the time elapsed between any two transitions.

Fact. (again without proof)
The continuous-time Markov chain $(X_t,\ t \in \mathbb{R}_+)$ is completely characterized by the parameters $(\nu_i)_{i \in S}$ (= the rates at which the chain leaves states) and $(\widehat{q}_{ij})_{i,j \in S}$ (= the transition probabilities of the embedded discrete-time chain).

From this, we also deduce the following (by an approximate reasoning):

$$p_{ii}(\Delta t) = \mathbb{P}(X_{\Delta t} = i | X_0 = i) \simeq \mathbb{P}(T_1 > \Delta t | X_0 = i) = e^{-\nu_i \Delta t} = 1 - \nu_i \Delta t + o(\Delta t) \quad (16)$$

$$\begin{aligned} p_{ij}(\Delta t) &= \mathbb{P}(X_{\Delta t} = j | X_0 = i) \simeq \mathbb{P}(X_{T_1} = j, T_1 \leq \Delta t | X_0 = i) \simeq \widehat{q}_{ij}\,(1 - e^{-\nu_i \Delta t}) \\ &= \widehat{q}_{ij} \nu_i \Delta t + o(\Delta t) \end{aligned} \quad (17)$$

Let us therefore define a new matrix Q as follows:

$$q_{ii} = -\nu_i \quad \text{and} \quad q_{ij} = \nu_i\, \widehat{q}_{ij}, \quad j \neq i$$

Then $|q_{ii}| = \nu_i$ represents the rate at which the chain leaves state i and $q_{ij} = \nu_i\, \widehat{q}_{ij}$ represents the rate at which the chain transits from state i to state j. Notice also that

$$\sum_{j \in S} q_{ij} = q_{ii} + \sum_{j \neq i} q_{ij} = -\nu_i + \nu_i \left(\sum_{j \neq i} \widehat{q}_{ij} \right) = 0, \quad \forall i \in S$$

Finally, we deduce from equations (16) and (17) that

$$P(\Delta t) = I + Q \Delta t + o(\Delta t)$$

The matrix Q characterizes therefore the short-term behavior of the continuous-time Markov chain X. It is therefore called the *infinitesimal generator* of X.

2.2.4 Kolmogorov equations

Proposition 2.14. (Kolmogorov equation: version 1)

$$\frac{d\pi_j(t)}{dt} = \sum_{i \in S} \pi_i(t)\, q_{ij}, \quad \forall i, j \in S, \quad \forall t \in \mathbb{R}_+$$

or in matrix form: $\dfrac{d\pi}{dt}(t) = \pi(t)\, Q$.

Dr. Mahesh Bundele
B.E., M.E., Ph.D.
Director
Poornima College of Engineering
ISI-6, RIICO Institutional Area
Sitapura, JAIPUR

Proof.

$$\pi_j(t+\Delta t) = \sum_{i\in S} \pi_i(t)\, p_{ij}\,(\Delta t) = \pi_j(t)\, p_{jj}(\Delta t) + \sum_{i\neq j} \pi_i(t)\, p_{ij}(\Delta t)$$
$$= \pi_j(t) + \sum_{i\in S} \pi_i(t)\, (q_{ij}\Delta t + o(\Delta t))$$

where the last equality is obtained using equations (16) and (17). Therefore,

$$\frac{\pi_j(t+\Delta t) - \pi_j(t)}{\Delta t} = \sum_{i\in S} \pi_i(t)\, q_{ij} + \frac{o(\Delta t)}{\Delta t}$$

so taking the limit $\Delta t \to 0$, we obtain (watch out that a technical detail is missing here)

$$\frac{d\pi_j}{dt}(t) = \lim_{\Delta t\to 0} \frac{\pi_j(t+\Delta t) - \pi_j(t)}{\Delta t} = \sum_{i\in S} \pi_i(t)\, q_{ij}$$

□

Proposition 2.15. (Kolmogorov equation: version 2, "forward" and "backward")

$$\frac{dp_{ij}}{dt}(t) = \sum_{k\in S} p_{ik}(t)\, q_{kj} = \sum_{k\in S} q_{ik}\, p_{kj}(t), \quad \forall i,j \in S, \quad \forall t \in \mathbb{R}_+$$

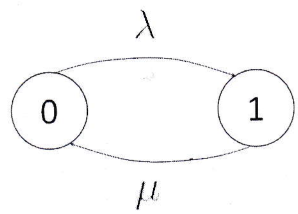
or in matrix form: $\dfrac{dP}{dt}(t) = P(t)\, Q = Q\, P(t).$

We skip the proof: it follows the same lines as above. The only difference is that the Chapman-Kolomogorov equation is used here:

$$p_{ij}(t+\Delta t) = \sum_{k\in S} p_{ik}(t)\, p_{kj}(\Delta t) = \sum_{k\in S} p_{ik}(\Delta t)\, p_{kj}(t)$$

Example 2.16. (two-state continuous-time Markov chain)
Let us consider the continuous-time Markov chain with state space $S = \{0, 1\}$ and infinitesimal generator

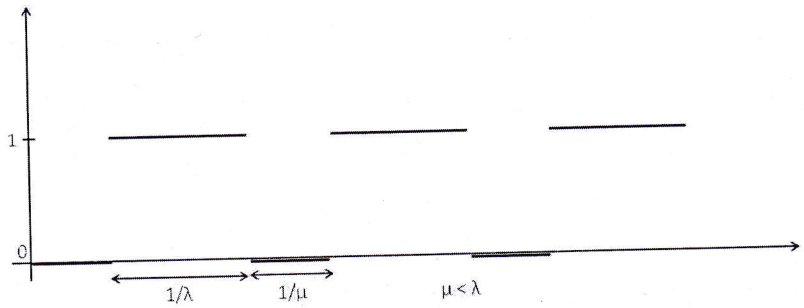
$$Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$



The embedded discrete-time Markov chain has the following transition matrix:

$$\widehat{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the average waiting times are $\dfrac{1}{\lambda}$ in state 0 and $\dfrac{1}{\mu}$ in state 1.



The Kolmogorov equation (version 1) reads in this case

$$\left(\frac{d\pi_0}{dt}, \frac{d\pi_1}{dt}\right) = (\pi_0, \pi_1) \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

Solving this ordinary differential equation, we obtain

$$\pi_0(t) = \frac{\mu}{\lambda+\mu} + \left(\pi_0(0) - \frac{\mu}{\lambda+\mu}\right) e^{-(\lambda+\mu)t}$$

and

$$\pi_1(t) = \frac{\lambda}{\lambda+\mu} + \left(\pi_1(0) - \frac{\lambda}{\lambda+\mu}\right) e^{-(\lambda+\mu)t}$$

2.2.5 Classification of states

As in the discrete-time case, let us introduce some definitions.

* Two states i and j *communicate* if $p_{ij}(t) > 0$ and $p_{ji}(t) > 0$ for some $t \geq 0$.

* The chain is said to be *irreducible* if all states communicate.

* Fact: if $p_{ij}(t) > 0$ for *some* $t > 0$, then $p_{ij}(t) > 0$ for *all* $t > 0$, so there is no notion of periodicity here.

* Let R_i be the *first return time* to state i: $R_i = \inf\{t > T_1 : X_t = i\}$.

* A state i is said to be *recurrent* if $f_i = \mathbb{P}(R_i < \infty | X_0 = i) = 1$ and *transient* otherwise.

* Moreover, if a state i is recurrent, then it is *positive recurrent* if $\mathbb{E}(R_i | X_0 = i) < \infty$ and *null recurrent* otherwise.

Remarks.
* The continuous-time Markov chain X and its embedded discrete-time Markov chain $\widehat{X}$ share all the above properties (except for periodicity).
* X is said to be *ergodic* if it is irreducible and positive recurrent (and as before, in a finite-state chain, irreducible implies positive recurrent).

2.2.6 Stationary and limiting distributions

The following theorem is the equivalent of Corollary 1.13 for discrete-time Markov chains.

Theorem 2.17. Let X be an ergodic continuous-time Markov chain. Then it admits a unique stationary distribution, i.e. a distribution π^* such that

$$\pi^* P(t) = \pi^*, \quad \forall t \in \mathbb{R}_+ \tag{18}$$

Moreover, this distribution is a limiting distribution, i.e. for any initial distribution $\pi(0)$, we have

$$\lim_{t\to\infty} \pi(t) = \pi^*$$

Remark.
Equation (18) is not so easy to solve in general, but it can be shown to be equivalent (modulo a technical assumption) to the much nicer equation

$$\pi^* Q = 0, \quad \text{i.e.} \quad \sum_{i\in S} \pi_i^* \, q_{ij} = 0, \quad \forall j \in S \tag{19}$$

Here is the main proof idea in one direction: if π^* satisfies (18), then $\pi^*(P(t) - P(0)) = 0$, $\forall t \in \mathbb{R}_+$. So

$$\lim_{t\to 0} \pi^* \left(\frac{P(t) - P(0)}{t} \right) = 0$$

which in turn implies (and here comes the technical detail that we skip) that $\pi^* \dfrac{dP}{dt}(0) = 0$, i.e. $\pi^* Q = 0$.

Example 2.18. (two-state continuous-time Markov chain)
Turning back to the two-state continuous-time Markov chain of Example 2.16 (which is ergodic), we need to solve

$$(\pi_0^*, \pi_1^*) \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} = 0$$

which leads to $(\pi_0^*, \pi_1^*) = \left(\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu} \right)$. Notice that this result could also have been obtained by taking the limit $t \to \infty$ in the expression obtained for $(\pi_0(t), \pi_1(t))$.

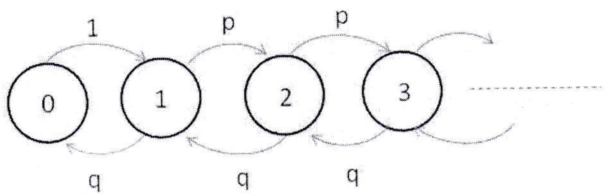
Example 2.19. (birth and death process)
Let $(X_t,\ t \in \mathbb{R}_+)$ be continuous-time Markov chain with state space $S = \mathbb{N}$ and infinitesimal generator

$$q_{0j} = \begin{cases} -\lambda & \text{if } j = 0 \\ \lambda & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad q_{ij} = \begin{cases} \mu & \text{if } j = i-1 \\ -\lambda-\mu & \text{if } j = i \\ \lambda & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i \geq 1$$

So $\nu_0 = \lambda$, and $\nu_i = \lambda + \mu$ for $i \geq 1$. Moreover, the embedded discrete-time Markov chain has the following transition probabilities:

$$\widehat{q}_{01} = \frac{q_{01}}{\nu_0} = 1, \quad \widehat{q}_{i,i+1} = \frac{q_{i,i+1}}{\nu_i} = \frac{\lambda}{\lambda+\mu}, \quad \widehat{q}_{i,i-1} = \frac{q_{i,i-1}}{\nu_i} = \frac{\mu}{\lambda+\mu}$$

corresponding to the transition graph



where $p = \frac{\lambda}{\lambda+\mu}$ and $q = 1 - p = \frac{\mu}{\lambda+\mu}$, i.e. this chain is a random walk on $\mathbb{N}$.

Let us now look for the stationary distribution of the continuous-time Markov chain, if it exists.

* If $\lambda, \mu > 0$, then the chain is irreducible.
* If $\lambda \geq \mu$, i.e. $p \geq q$, then the chain is either transient or null recurrent. so there does not exist a stationary distribution.
* If on the contrary $\lambda < \mu$, i.e. $p < q$, then the chain is positive recurrent, and solving the equation $\pi^* Q = 0$ in this case leads to

$$\pi_0\, q_{00} + \pi_1\, q_{10} = 0, \quad \pi_{i-1}\, q_{i-1,i} + \pi_i\, q_{ii} + \pi_{i+1}\, q_{i+1,i} = 0$$
$$-\lambda\, \pi_0 + \mu\, \pi_1 = 0, \quad \lambda\, \pi_{i-1} - (\lambda+\mu)\, \pi_i + \mu\, \pi_{i+1} = 0$$

So

$$\pi_1 = \frac{\lambda}{\mu}\pi_0, \quad \pi_2 = \frac{1}{\mu}\left((\lambda+\mu)\pi_1 - \lambda\pi_0\right) = \left(\frac{\lambda}{\mu}\right)^2 \pi_0$$

and by induction,

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \pi_0, \quad \pi_0\left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2} + \ldots\right) = 1$$

i.e., finally,

$$\pi_k^* = \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right), \quad k \in \mathbb{N}$$

This concludes these short lecture notes on Markov chains.